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HYPER LIE-SANTILLI ADMISSIBILITY[1]

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Abstract

The Lie-Santilli Admissibility introduced by R. M. Santilli, can be transferred in the hyperstructure theory. We present the Hyper-Lie-Santilli's Admissibility in the general case. This class of hyperstructures is very large, so special cases can be used in applications.

AMS Subject Classification: 20N20, 16Y99
Key words: *hyperstructure, H_v-structure, hope, hypernumbers, isonumbers, Lie-Santilli's admissibility.*

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1. INTRODUCTION

The largest class of hyperstructures is the class of *H_v-structures* introduced in 1990 [16]. The H_v-structures satisfy the weak axioms where the non-empty intersection replaces the equality. Some basic definitions and results:

Algebraic hyperstructure is called a set H equipped with at least one *hyperoperation* (abbreviation: ***hyperoperation=hope***) $\cdot: H\times H\to P(H)-\{\varnothing\}$. Abbreviate by *WASS* the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by *COW* the *weak commutativity*: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$. The hyperstructure $(H,\cdot)$ is called *H_v-semigroup* if it is *WASS,* it is called ***H_v-group*** if it is reproductive H_v-semigroup, i.e. $xH=Hx=H$, $\forall x\in H$.

Motivation. The quotient of a group by invariant subgroup, is a group. F. Marty (1934): the quotient of a group by a subgroup is a hypergroup. Now the quotient of a group by a partition ($\cong$ any equivalence) is an H_v-group.

In an H_v-semigroup the *powers* are defined by: $h^1=\{h\}$, $h^2=h\cdot h,\ldots$, $h^n=h\circ h\circ\ldots\circ h$, where $(\circ)$ is the *n-ary circle hope,* i.e. we take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H,\cdot)$ is *cyclic of period s*, if there is an h, called *generator*, and a natural s, the minimum: $H=h^1\cup h^2\cup\ldots\cup h^s$. If there is an h and s, the minimum: $H=h^s$, then $(H,\cdot)$, is called *single-power cyclic of period s.*

$(R,+,\cdot)$ is called ***H_v-ring*** if $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\quad \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be H_v-ring, $(M,+)$ be a COW H_v-group and there exists a hope

$$\cdot:\ R\times M\to P(M):\ (a,x)\to ax$$

such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\ (a+b)x\cap(ax+bx)\neq\varnothing,\ (ab)x\cap a(bx)\neq\varnothing,$$

then M is called an *H_v-module* over F. In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the ***H_v-vector space*** is defined.

For more definitions and applications on H_v-structures one can see in books and papers as [1],[3],[4],[5],[16],[17],[18],[19],[20],[21],[29],[31],[32].

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups on the same H, the hope $(\cdot)$ is *smaller* than the $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$f \in Aut(H,*)$ such that $xy \subset f(x*y)$, $\forall x,y \in H$.

We write $\cdot \leq^*$ and $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then $(H,*)$ is called *H_b-structure.*

The Little Theorem. Greater hopes than WASS or COW, are WASS or COW, respectively.

This Theorem leads to a partial order on H_v-structures and to posets.

Let $(H,\cdot)$ be hypergroupoid. We *remove* $h \in H$, if we take the restriction of $(\cdot)$ in the set $H-\{h\}$. $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h} \in H$ *merges* with $h \in H$, if we take as product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class.

The main tool in hyperstructures is the *fundamental relation.* M. Koscas 1970 defined in hypergroups the relation β and its transitive closure β^*. This connects hyperstructures with the classical structures and is defined in H_v-groups as well. T. Vougiouklis [12], [14], [16], [18], [19], [20] introduced the γ^* and ε^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations, *fundamental.*

Definition 1.1 The ***fundamental relations*** β^*, γ^* and ε^* are defined in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.

Specifying the above motivation we remark that: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the *fundamental.*

The way to find the fundamental classes is given by analogous theorems to:

Theorem Let $(H,\cdot)$ an H_v-group and U the set of all finite products of elements of H. Take the relation β in H by setting $x\beta y$ iff $\{x,y\} \subset u$, where $u \in U$. Then β^* is the transitive closure of β.

An element is called ***single*** if its fundamental class is singleton [18].

More general structures can be defined by using the fundamental structures. An application is the general hyperfield [16], [18].

Definition 1.2 An H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if R/γ^* is a field.

Definition 1.3 [26], [27], [30], [13] Let $(L,+)$ be an H_v-vector space over the H_v-field $(F,+,\cdot)$, $\varphi: F \to F/\gamma^*$, the canonical map and $\omega_F=\{x \in F: \varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Let ω_L be the core of the canonical

map $\varphi': L \to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Consider the *bracket hope*:

$$[\,,\,]: L\times L \to P(L): (x,y) \to [x,y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$[\lambda_1 x_1+\lambda_2 x_2,y] \cap (\lambda_1[x_1,y]+\lambda_2[x_2,y]) \neq \varnothing$

$[x,\lambda_1 y_1+\lambda_2 y] \cap (\lambda_1[x,y_1]+\lambda_2[x,y_2]) \neq \varnothing$, $\forall x,x_1,x_2,y,y_1,y_2 \in L$ and $\lambda_1,\lambda_2 \in F$

(L2) $[x,x] \cap \omega_L \neq \varnothing$, $\forall x \in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]]) \cap \omega_L \neq \varnothing$, $\forall x,y \in L$

Definition 1.4 [22]. The H_v-semigroup $(H,\cdot)$ is ***h/v-group*** if H/β^* is a group.

The h/v-groups are more general than the H_v-groups since in h/v-groups the *reproductivity of classes* is valid, i.e. $x\sigma(y)=\sigma(xy)=\sigma(x)y$, $\forall x,y \in H$. Similarly the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc, are defined.

The *uniting elements* method, introduced by Corsini & Vougiouklis [2] in 1989, is the following: Let $\boldsymbol{G}$ be a structure and d, a property which is not valid. Suppose d is described by a set of equations; then, take the partition in $\boldsymbol{G}$ for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient $\boldsymbol{G}/d$ is an H_v-structure. Quotient out $\boldsymbol{G}/d$ by β^*, the structure $(\boldsymbol{G}/d)\beta^*$ for which the property d is valid, is obtained.

Theorem 1.5 [18]. Let $(\boldsymbol{R},+,\cdot)$ be ring, and $F=\{f_1,\ldots,f_m,f_{m+1},\ldots,f_{m+n}\}$ be a system of equations on $\boldsymbol{R}$ consisting of two subsystems $F_m=\{f_1,\ldots,f_m\}$ and $F_n=\{f_{m+1},\ldots,f_{m+n}\}$. Let σ, σ_m be the relations defined by uniting elements using the systems F and F_m respectively, and σ_n is defined by induced equations of F_n on the ring $\boldsymbol{R}_m=(\boldsymbol{R}/\sigma_m)/\gamma^*$. Then $(\boldsymbol{R}/\sigma)/\gamma^* \cong (\boldsymbol{R}_m/\sigma_n)/\gamma^*$.

Definition 1.6 [17] An H_v-structure is ***very thin*** if all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one.

From the very thin hopes the *Attach Construction* is obtained [20]: Let $(H,\cdot)$ be an H_V-semigroup and $v \notin H$. Extend $(\cdot)$ into $\underline{H}=H\cup\{v\}$ by: $x\cdot v=v\cdot x=v, \forall x \in H$, and $v\cdot v=H$. The $(\underline{H},\cdot)$ is an H_V-group, where $(\underline{H},\cdot)/\beta^* \cong Z_2$ and v is single.

Definition 1.7 [23] Let $(G,\cdot)$ be groupoid (resp., hypergroupoid), $f:G \to G$ be a map. We define a hope (∂), called *theta-hope,* we write *∂-hope*, on G as follows

$x\partial y = \{f(x)\cdot y, x\cdot f(y)\}$, $\forall x,y\in G$. (resp. $x\partial y = (f(x)\cdot y)\cup(x\cdot f(y))$, $\forall x,y\in G$)

If $(\cdot)$ is commutative or *COW*, then ∂ is commutative or *COW*, resp. If $(G,\cdot)$ is a groupoid and $f:G\to P(G)-\{\varnothing\}$, multivalued. The ∂-hope is defined by:

$$x\partial y=(f(x)\cdot y)\cup(x\cdot f(y)),\ \forall x,y\in G.$$

Motivation for ∂-hopes is the *derivative* where only the product of functions can be used. Basic property: if $(G,\cdot)$ is semigroup then $\forall f$, the ∂-hope is *WASS*.

Examples.

(a) In integers $(\boldsymbol{Z},+,\cdot)$ fix $n\neq 0$, a natural number. Consider the map f such that $f(0)=n$ and $f(x)=x$, $\forall x\in\boldsymbol{Z}-\{0\}$. Then $(Z,\partial_+,\partial\cdot)$, where ∂_+ and $\partial\cdot$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with $(\boldsymbol{Z},\partial_+,\partial\cdot)/\gamma^* \cong \boldsymbol{Z}_n$.

(b) In $(\boldsymbol{Z},+,\cdot)$ with $n\neq 0$, take f such that $f(n)=0$ and $f(x)=x$, $\forall x\in\boldsymbol{Z}-\{n\}$. Then $(\boldsymbol{Z},\partial_+,\partial\cdot)$ is an H_v-ring, moreover, $(\boldsymbol{Z},\partial_+,\partial\cdot)/\gamma^* \cong \boldsymbol{Z}_n$.

Special case of the above is for n=p, prime, then $(\boldsymbol{Z},\partial_+,\partial\cdot)$ is an H_v-field.

One can see ∂-hopes in H_v-vector spaces and H_v-Lie algebras [23],[31],[32]:

Theorem. Let $(\boldsymbol{V},+,\cdot)$ be an algebra over the field $(F,+,\cdot)$ and $f:\boldsymbol{V}\to\boldsymbol{V}$ be a map. Consider the ∂-hope defined only on the multiplication of the vectors $(\cdot)$, then $(\boldsymbol{V},+,\partial)$ is an H_v-algebra over F, where the related properties are weak.

Theorem 1.8 Let $(\boldsymbol{A},+,\cdot)$ be an algebra over the field F and $f:\boldsymbol{A}\to\boldsymbol{A}$ be a map. The *∂-hope on the Lie bracket* $[x,y]=xy-yx$, is defined as follows

$$x\partial y = \{f(x)y-f(y)x,\ f(x)y-yf(x),\ xf(y)-f(y)x,\ xf(y)-yf(x)\}.$$

$(\boldsymbol{A},+,\partial)$ is an H_v-algebra on F, the weak anti-commutativity and the inclusion linearity is valid.

Combining the uniting elements procedure with the enlarging, we obtain:

Theorem. In the ring $(\boldsymbol{Z}_n,+,\cdot)$, with n=ms we enlarge the multiplication only the product $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then $(\boldsymbol{Z}_n,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_m,+,\cdot)$.

We can enlarge other products as well, as $2\cdot m$ by setting $2\otimes m=\{2,m+2\}$, then the result remains the same. In this case 0 and 1 remain scalars.

Corollary. In $(\boldsymbol{Z}_n,+,\cdot)$, with n=ps where p prime, we enlarge only the $0\cdot p$ by $0\otimes p=\{0,p\}$ and the rest remain the same. Then $(\boldsymbol{Z}_n,+,\otimes)$ is very thin H_v-field.

A large class of H_v-structures, based on classical, is the following [15], [27]:

Definition 1.9 Let $(G,\cdot)$ be groupoid, then $\forall P\subset G$, $P\neq\emptyset$, the following ***P-hopes*** are defined:

$\underline{P}$: $x\underline{P}y=(xP)y\cup x(Py)$, $\underline{P_r}$: $x\underline{P_r}y=(xy)P\cup x(yP)$, $\underline{P_l}$: $x\underline{P_l}y=(Px)y\cup P(xy)$, $\forall x,y\in G$

The $(G,\underline{P})$, $(G,\underline{P_r})$ and $(G,\underline{P_l})$ are called *P-hyperstructures*. If $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup

A *generalization of P-hopes*, used in Santilli's isotheory, is the following [6], [11], [29]: Let $(G,\cdot)$ be abelian group and P with $\#P>1$. We define the hope $(\times_P)$ as follows:

$$x\times_P y = \begin{cases} x\cdot P\cdot y=\{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } c\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G,\times_P)$ is abelian H_v-group.

We present now the *small non-degenerate h/v-fields* on $(\boldsymbol{Z}_n,+,\cdot)$ which satisfy the following conditions, appropriate in Santilli's iso-theory:

1. *very thin minimal,*
2. *COW (non-commutative),*
3. *they have the elements 0 and 1, scalars,*
4. *if an element has inverse element, this is unique.*

Remark that last condition means than we cannot enlarge the result if it is 1 and we cannot put 1 in any other enlargement. Moreover we study only the upper triangular cases since the corresponding under triangular cases are isomorphic due to the commutativity of rings.

Theorem 1.10 All multiplicative h/v-fields defined on $(\boldsymbol{Z}_4,+,\cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:

The only product which is set is $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$.

The fundamental classes are $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have

$$(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot).$$

Theorem 1.11 [31] All multiplicative h/v-fields on $(\boldsymbol{Z}_6,+,\cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:

We have the only one hyperproduct,

(I) $2\otimes 3=\{0,3\}$ or $2\otimes 4=\{2,5\}$
$3\otimes 4=\{0,3\}$ or $3\otimes 5=\{0,3\}$ or $4\otimes 5=\{2,5\}$

Fundamental classes: [0]={0,3}, [1]={1,4}, [2]={2,5} and

$$(\boldsymbol{Z}_6,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_3,+,\cdot).$$

(II) $2\otimes 3=\{0,2\}$ or $2\otimes 3=\{0,4\}$ or $2\otimes 4=\{0,2\}$ or $2\otimes 4=\{2,4\}$ or $2\otimes 5=\{0,4\}$ or $2\otimes 5=\{2,4\}$ or $3\otimes 4=\{0,2\}$ or $3\otimes 4=\{0,4\}$ or $3\otimes 5=\{3,5\}$ or $4\otimes 5=\{0,2\}$ or $4\otimes 5=\{2,4\}$.

Fundamental classes: [0]={0,2,4}, [1]={1,3,5} and

$$(\boldsymbol{Z}_6,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_2,+,\cdot).$$

Example. In the case of the h/v-field $(\boldsymbol{Z}_6,+,\otimes)$ where the only one hyperproduct is $3\otimes 4=\{0,3\}$ we consider the h/v-matrices of type

$$\underline{i}=E_{11}+iE_{12}+4E_{22}, \text{ where } i=0,1,2,3,4,5,$$

then the multiplicative table of the hyperproduct of those h/v-matrices is

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$
$\underline{3}$	$\underline{0},\underline{3}$	$\underline{1},\underline{4}$	$\underline{2},\underline{5}$	$\underline{0},\underline{3}$	$\underline{1},\underline{4}$	$\underline{2},\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

$(\boldsymbol{Z}_6,\otimes)$ is a cyclic h/v-group.

2. THE ISO-HYPERNUMBERS

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems [8], [9]. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 [10] and they are called *e-hyperfields.*

Definition 2.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. We assume that $\forall x$, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 2.2 A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid:

$(F,+)$ is an abelian group with the additive unit 0,

$(\cdot)$ is WASS,

$(\cdot)$ is weak distributive with respect to $(+)$,

0 is absorbing element: $0\cdot x=x\cdot 0=0$,

$\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and

$\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 2.3 *Main e-Construction* [25]. Let $(G,\cdot)$, be a group, e the unit, then we define in G, a large number of hopes $(Ä)$ as follows:

$$xÄy = \{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\}, \text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$(G,Ä)$ is an H_b-group which contains the $(G,\cdot)$. $(G,Ä)$ is e-hypergroup. Moreover, if $\forall x,y$ such that $xÄy=xy=e$, then $(G,Ä)$ is a strong e-hypergroup.

The main e-construction gives an extremely large number of e-hopes.

Example. Consider the quaternions $\boldsymbol{Q}=\{1,-1, i,-i, j,-j, k,-k\}$, with $i^2=j^2=-1$, $ij=-ji=k$, and denote $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$. We define a lot of hopes $(*)$ by enlarging few products. For example, $(-1)*k=\underline{k}$, $k*i=\underline{j}$ and $i*j=\underline{k}$. Then the hyperstructure $(Q,*)$ is a strong e-hypergroup.

In Santilli's iso-theory [9], on a field $F=(F,+,\cdot)$, *a general isofield* $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}} \qquad \text{(i)}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of Fé, i.e.

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\times\hat{1}=\hat{a},\ \forall\hat{a}\in\hat{\boldsymbol{F}} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

Construction 2.4 *General enlargement.* On a field $F=(F,+,\cdot)$ and the isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ we replace in the results of the iso-product

$$\hat{a}\,\hat{\times}\,b = \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements, containing $\hat{T}$, $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$, where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}$, for all hyperproducts $\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore the new iso-hope is

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{H}_{ab}\times\hat{b} = \hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(iii)}$$

$\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ becomes *isoH$_v$-field.* The elements of F are called *isoH$_v$-numbers* or *isonumbers.*

More important hopes, are the ones where only for few ordered pairs $(\hat{a},\hat{b})$ the result is enlarged. Even more in the special case is if there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{F}$$

then we obtain the *very thin isoH$_v$-field* [10], [11], [24], [28], [29].

Construction 2.5 *The P-hope.* Consider an isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ with $\hat{a}=a\times\hat{1}$, the isonumbers, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is the iso-multiplication

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{F}$$

Take $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, with $\hat{p}_1,\ldots,\hat{p}_s\in\hat{F}-\{\hat{0},\hat{1}\}$, we define the *iso-P-H$_v$-field,* $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times}_P)$ where the hope $\hat{\times}_P$ is defined as follows:

$$\hat{a}\,\hat{\times}_P\,\hat{b} := \begin{cases} \hat{a}\times\hat{P}^{\wedge}\times\hat{b} = \{\hat{a}\times\hat{h}^{\wedge}\times\hat{b}\,|\,\hat{h}^{\wedge}\in\hat{P}^{\wedge}\} & \text{if } \hat{a}\neq\hat{1} \text{ and } \hat{b}\neq\hat{1} \\ \hat{a}\times\hat{T}^{\wedge}\times\hat{b} & \text{if } \hat{a}=\hat{1} \text{ or } \hat{b}=\hat{1} \end{cases} \qquad \text{(iv)}$$

The elements of $\hat{F}$ are called *iso-P-H$_v$-numbers.* The inverses in iso-P-H$_v$-fields, are not necessarily unique.

Remarkable is when $\hat{P}=\{\hat{T},\hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$.

Non degenerate examples on the above constructions on the small finite field $(Z_7,+,\cdot)$. In the $\hat{Z}_7=\hat{Z}_7(\hat{+},\hat{\times})$, where we denote $\hat{Z}_7=\{\underline{\hat{0}},\underline{\hat{1}},\underline{\hat{2}},\underline{\hat{3}},\underline{\hat{4}},\underline{\hat{5}},\underline{\hat{6}}\}$, the

WASS multiplicative hope is described by only one enlargement $\hat{\underline{2}} \hat{\times} \hat{\underline{6}} = \{\hat{\underline{2}},\hat{\underline{5}}\}$

3. HYPERMATRIX REPRESENTATIONS

H_v-structures are used in Representation (abbreviate by ***rep***) Theory. Reps of H_v-groups can be considered either by generalized permutations or by H_v-matrices [12], [14], [18], [21], [31], [32]. The reps by generalized permutations can be achieved by using left or right translations. We present here the hypermatrix rep in H_v-structures and there exist the completely analogous theory for the h/v-structures.

Definitions 3.1 ***H_v-matrix*** is a matrix with entries elements of an H_v-field (or an H_v-ring). The hyperproduct of two H_v-matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$, of type $m\times n$ and $n\times r$ respectively, is defined, in the usual manner,

$$\mathbf{A}\cdot\mathbf{B} = (a_{ij})\cdot(b_{ij}) = \{\ \mathbf{C}=(c_{ij}) \mid c_{ij}\in\oplus\Sigma a_{ik}\cdot b_{kj}\ \},$$

and it is a set of $m\times r$ H_v-matrices. The sum of products of elements of the H_v-field is obtained by the *n-ary circle hope* on the hyperaddition. The hyperproduct is not necessarily WASS.

Definitions 3.2 Let $(H,\cdot)$ be H_v-group, take an H_v-field F, a set $M_F=\{(a_{ij}) \mid a_{ij}\in F\}$ and a map

$$\mathbf{T}: H \to M_F: h \mapsto \mathbf{T}(h) \text{ such that } \mathbf{T}(h_1h_2)\cap\mathbf{T}(h_1)\mathbf{T}(h_2) \neq \varnothing,\ \forall h_1,h_2\in H.$$

T is called ***H_v-matrix rep***. If the $\mathbf{T}(h_1h_2)\subset\mathbf{T}(h_1)\mathbf{T}(h_2)$, $\forall h_1,h_2\in H$, then **T** is an *inclusion rep*. If $\mathbf{T}(h_1h_2)=\mathbf{T}(h_1)\mathbf{T}(h_2)=\{\mathbf{T}(h) \mid h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then **T** is a *good rep* and if **T** is one to one and good then it is a *faithful rep.*

The main theorem of reps is the following [12], [14], [18]:

Theorem 3.3 A necessary condition in order to have an inclusion rep **T** of an H_v-group $(H,\cdot)$ by $n\times n$ H_v-matrices over the H_v-field $(F,+,\cdot)$ is the following: For all classes $\beta^*(x)$, $x\in H$ there must exist $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$\mathbf{T}(\beta^*(a)) \subset \{A=(a'_{ij}) \mid a'_{ij}\in \gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}$$

The inclusion reps $\mathbf{T}:H\to M_R:a\mapsto \mathbf{T}(a)=(a_{ij})$ induce a homomorphic rep $\mathbf{T}^*$ of H/β^* on F/γ^* by setting $\mathbf{T}^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of $\mathbf{T}^*(\beta^*(a))$. $\mathbf{T}^*$ is called *fundamental induced rep* of **T**.

Let $tr_\varphi(\mathbf{T}(x))=\gamma^*(\mathbf{T}(x_{ii}))$ be the fundamental trace, then is called *fundamental character*, the map

$$X_T: H\to R/\gamma^*: x\mapsto X_T(x)= tr_\varphi(T(x))= trT^*(x)$$

Some hopes 'like' P-hopes are the following [18], [21], [31]:

Definition 3.4 Let $M=M_{m,n}$ be the m×n matrices over R and $P=\{P_i:i\in I\}\subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on M as follows

$$\underline{P}: M\times M\to P(M): (A,B)\ A\underline{P}B=\{AP^t_iB: i\in I\}\subseteq M$$

where P^t denotes the transpose of P.

The hope $\underline{P}$ is a bilinear map, strong associative and inclusion distributive:

$$A\underline{P}(B+C)\subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C\in M$$

Therefore, $(M,+,\underline{P})$ defines a multiplicative hyperring on non-square matrices.

Let $\mathbf{M}=\mathbf{M}_{m,n}$ be a module of m×n matrices over R and let us take sets

$$\mathbf{S}=\{s_k:k\in K\}\subseteq R,\ \mathbf{Q}=\{Q_i:j\in J\}\subseteq\mathbf{M},\ \mathbf{P}=\{P_i:i\in I\}\subseteq\mathbf{M}.$$

Define three hopes as follows

$$\underline{S}: R\times\boldsymbol{M}\to\boldsymbol{P}(\boldsymbol{M}): (r,A)\to r\underline{S}A=\{(rs_k)A: k\in K\}\subseteq\boldsymbol{M}$$

$$\underline{Q_+}: \boldsymbol{M}\times\boldsymbol{M}\to\boldsymbol{P}(\boldsymbol{M}): (A,B)\to A\underline{Q_+}B=\{A+Q_j+B: j\in J\}\subseteq\boldsymbol{M}$$

$$\underline{P}: \boldsymbol{M}\times\boldsymbol{M}\to\boldsymbol{P}(\boldsymbol{M}): (A,B)\to A\underline{P}B=\{AP^t_iB: i\in I\}\subseteq\boldsymbol{M}$$

Then $(\boldsymbol{M},\underline{S},\underline{Q_+},\underline{P})$ is hyperalgebra on $\boldsymbol{R}$ called *general matrix P-hyperalgebra.*

New definitions on this topic appear [31], [32]:

Definition 3.5 Let $A=(a_{ij})$, $B=(b_{ij})\in\mathbf{M}_{m,n}$, we call (A,B) a *unitize pair* of matrices if $A^tB=I_n$, where I_n denotes the n×n unit matrix.

The following theorem can be applied in the classical theory [32].

Theorem. If m<n, then there is no unitize pair.

An important hope on non-square ordinary matrices can be defined [33]:

Definitions 3.6 Let $A=(a_{ij})\in M_{m,n}$ be m×n matrix and s,t∈N naturals $1\le s\le m$, $1\le t\le n$. Define the characteristic-like map $\underline{cst}$ from $M_{m,n}$ to $M_{s,t}$ by corresponding to A, the matrix $A\underline{cst}=(a_{ij})$ where $1\le i\le s$, $1\le j\le t$, called *cut-projection* of type $\underline{st}$.

Let $A=(a_{ij})\in M_{m,n}$ and s,t∈N, $1\le s\le m$, $1\le t\le n$. We define the *helix-projection* of type $\underline{st}$ by

$$\underline{st}: M_{m,n}\to M_{s,t}: A\to A\underline{st}=(\underline{a}_{ij}),$$

where

$$\underline{a}_{ij} = \{a_{i+\kappa s, j+\lambda t} \mid 1 \le i \le s,\ 1 \le j \le t. \text{ and } \kappa,\lambda \in N,\ i+\kappa s \le m,\ j+\lambda t \le n\}.$$

Definitions 3.7 (a) Let $A=(a_{ij}) \in M_{m*n}$ and $B=(b_{ij}) \in M_{u*v}$ with $s=\min(m,u)$, $t=\min(n,u)$. We define a hope, called *helix-addition*, as follows:

$$\oplus: M_{m*n} \times M_{u*v} \to P(M_{s*t}): (A,B) \to A \oplus B = A\underline{st} + B\underline{st} = (\underline{a}_{ij}) + (\underline{b}_{ij}) \subset M_{s*t},$$

where

$$(\underline{a}_{ij}) + (\underline{b}_{ij}) = \{(c_{ij}) = (a_{ij}+b_{ij}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}.$$

(b) Let $A=(a_{ij}) \in M_{m*n}$ and $B=(b_{ij}) \in M_{u*v}$ be matrices and $s=\min(n,u)$. We define a hope, called *helix-multiplication*, as follows:

$$\otimes: M_{m*n} \times M_{u*v} \to P(M_{m*v}): (A,B) \to A \otimes B = A\underline{ms} \cdot B\underline{sv} = (\underline{a}_{ij}) \cdot (\underline{b}_{ij}) \subset M_{m*v},$$

where

$$(\underline{a}_{ij}) \cdot (\underline{b}_{ij}) = \{(c_{ij}) = (\sum a_{it} b_{tj}) \mid a_{ij} \in \underline{a}_{ij} \text{ and } b_{ij} \in \underline{b}_{ij}\}.$$

In the set M_{m*n} the addition is an ordinary operation, thus we define the hyperproduct. Considering the Lie-bracket, the *helix-Lie Algebra* is defined [26], [29], is defined as well.

4. THE HYPER LIE-SANTILLI ADMISSIBILITY

The general definition of the Lie-Santilli admissibility is [9], [11], [29]:

Definition 4.1 Let $\boldsymbol{L}$ be an H_v-vector space over the H_v-field $(\boldsymbol{F},+,\cdot)$, $\varphi: \boldsymbol{F} \to \boldsymbol{F}/\gamma^*$, the canonical map and $\omega_F = \{x \in \boldsymbol{F}: \varphi(x) = 0\}$, where 0 is the zero of the fundamental field $\boldsymbol{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\varphi': \boldsymbol{L} \to \boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Take any two subsets $\boldsymbol{R},\boldsymbol{S} \subseteq \boldsymbol{L}$ then a ***Lie-Santilli admissible hyperalgebra*** is obtained by taking the Lie bracket, which is a hope:

$$[\ ,\ ]_{RS}: \boldsymbol{L} \times \boldsymbol{L} \to \boldsymbol{P}(\boldsymbol{L}): [x,y]_{RS} = x\boldsymbol{R}y - y\boldsymbol{S}x = \{xry - ysx \mid r \in \boldsymbol{R},\ s \in \boldsymbol{S}\}$$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) When only $\boldsymbol{S}$ is considered, then $[x,y]_S = xy - y\boldsymbol{S}x = \{xy - ysx \mid s \in \boldsymbol{S}\}$

(b) When only $\boldsymbol{R}$ is considered, then $[x,y]_R = x\boldsymbol{R}y - yx = \{xry - yx \mid r \in \boldsymbol{R}\}$

(c) When $\boldsymbol{R}=\{r_1,r_2\}$ and $\boldsymbol{S}=\{s_1,s_2\}$ then

$$[x,y]_{RS} = x\boldsymbol{R}y - y\boldsymbol{S}x = \{xr_1y - ys_1x,\ xr_1y - ys_2x,\ xr_2y - ys_1x,\ xr_2y - ys_2x\}.$$

(d) When $\boldsymbol{S}=\{s_1,s_2\}$ then $[x,y]_S = xy - y\boldsymbol{S}x = \{xy - ys_1x,\ xy - ys_2x\}$.

(e) When $\boldsymbol{R}=\{r_1,r_2\}$ then $[x,y]_R = x\boldsymbol{R}y - yx = \{xr_1y - yx,\ xr_2y - yx\}$.

(f) We have one case which is 'like' P-hope for any subset $\boldsymbol{S}\subseteq\boldsymbol{L}$:

$$[x,y]_S = \{xsy - ysx \mid s\in\boldsymbol{S}\}$$

Example 4.2 In the set of 2×2 traceless complex matrices we take the set of self inverse matrices

$$\boldsymbol{R} = \boldsymbol{S} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

Then, take two traceless matrices

$$\boldsymbol{X} = \begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix},\quad \boldsymbol{Y} = \begin{bmatrix} y_1 & y_2 \\ y_3 & -y_1 \end{bmatrix}$$

the Lie bracket is the following

$$[X,Y]_{RR} = X\boldsymbol{R}Y - Y\boldsymbol{R}X =$$

$$\left\{ \begin{bmatrix} x_2y_1+x_1y_3-x_1y_2-x_3y_1 & 0 \\ 0 & x_2y_1+x_1y_3-x_1y_2-x_3y_1 \end{bmatrix},\right.$$

$$\begin{bmatrix} x_2y_1+x_1y_3-x_1y_1+x_3y_2 & x_2y_2-x_1y_1-x_2y_1-x_1y_2 \\ -x_1y_1+x_3y_3-x_1y_3-x_3y_1 & -x_1y_2-x_3y_1-x_2y_3+x_1y_1 \end{bmatrix},$$

$$\left.\begin{bmatrix} x_1y_1-x_2y_3-x_1y_2-x_3y_1 & x_1y_2+x_2y_1-x_2y_2+x_1y_1 \\ x_2y_1+x_1y_3+x_1y_1-x_3y_3 & x_3y_2-x_1y_1+x_2y_1+x_1y_3 \end{bmatrix}, \begin{bmatrix} -x_2y_3+x_3y_2 & 0 \\ 0 & x_3y_2-x_2y_3 \end{bmatrix} \right\}$$

Therefore the result is a set of 4 matrices, where the last one is traceless.

The results can be simplified in special cases, for example in the above if we consider the upper triangular traceless matrices, we have

$$\boldsymbol{X} = \begin{bmatrix} x_1 & x_2 \\ 0 & -x_1 \end{bmatrix},\quad \boldsymbol{Y} = \begin{bmatrix} y_1 & y_2 \\ 0 & -y_1 \end{bmatrix}$$

then the Lie bracket is the following

$$[X,Y]_{RR} = X\boldsymbol{R}Y - Y\boldsymbol{R}X =$$

$$\left\{ \begin{bmatrix} x_2y_1-x_1y_2 & 0 \\ 0 & x_2y_1-x_1y_2 \end{bmatrix}, \begin{bmatrix} x_2y_1-x_1y_1 & x_2y_2-x_1y_1-x_2y_1-x_1y_2 \\ -x_1y_1 & -x_1y_2+x_1y_1 \end{bmatrix},\right.$$

$$\left. \begin{bmatrix} x_1y_1-x_1y_2 & x_1y_2+x_2y_1-x_2y_2+x_1y_1 \\ x_2y_1+x_1y_1 & -x_1y_1+x_2y_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Example 4.3 In the set of 3×3 traceless complex matrices of the form

$$X=\begin{pmatrix} x_1 & x_2 & x_3 \\ 0 & -x_1-1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R=\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

Then, if we denote

$$q_1 = -x_1y_2+x_2y_1+2x_2-2y_2, \qquad q_2 = -x_1y_2+x_2y_1+x_1y_1+y_1+2x_2-2y_2,$$

$$q_3 = -x_1y_2+x_2y_1-x_1y_1+x_1+2x_2-2y_2, \qquad q_4 = -x_1y_2+x_2y_1-x_1+y_1+2x_2-2y_2,$$

we obtain that the Lie bracket is the following

$$[X,Y]_{RR} = XRY-YRX = \left\{\begin{pmatrix} 0 & q_i & x_1y_3+x_3-x_3y_1-x_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : i=1,2,3,4\right\}$$

Remark that in special pairs (X,Y) we may have in Lie bracket less than 4 elements. For example, if we take $y_1=0$, then the result is one element.

We present the Lie-Santilli's admissibility on non-square matrices [6], [7], [11], [27], [30].

Construction 4.4 Let $(L=M_{m,n},+)$ be H_v-vector space of m×n hyper-matrices over the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of F/γ^*. Similarly, let ω_L be the core of the canonical map $\varphi':L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Take any two subsets $R,S\subseteq L$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hope:

$$[\,,\,]_{RS}:\ L\times L\to P(L):\ [x,y]_{RS}= xR^ty-yS^tx= \{xr^ty-ys^tx \mid r\in R,\ s\in S\}$$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) The case which $\forall S\subseteq L$: $[x,y]_S = \{xs^ty-ys^tx \mid s\in S\}$

(b) $R=\{r\}$, $\forall S\subseteq L$, then $[x,y]_{RS} = xr^ty-yS^tx = \{xr^ty-ys^tx \mid s\in S\}$

(c) $S=\{s\}$, $\forall R\subseteq L$, then $[x,y]_{RS} = xR^ty-ys^tx = \{xr^ty-ys^tx \mid r\in R\}$

(d) $R=\{r_1,r_2\}$ and $S=\{s_1,s_2\}$ then

$[x,y]_{RS} = xR^ty-yS^tx = \{xr_1^ty-ys_1^tx,\ xr_1^ty-ys_2^tx,\ xr_2^ty-ys_1^tx,\ xr_2^ty-ys_2^tx\}$

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CREATING NEW ALGEBRAIC HYPERSTRUCTURES FROM SANTILLI'S ISO & GENO-THEORIES[1]

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Abstract

In the quiver of hyperstructures Professor R. M. Santilli in early 90'es, tried to find algebraic structures in order to express his pioneer Lie-Santilli's Theory. Santilli's theory on 'isotopies' and 'genotopies', born in 1960's, desperately needs 'units e' on left or right, which are nowhere singular, symmetric, real-valued, positive-defined for n-dimensional matrices based on the so called isofields.These elements can be found in hyperstructure theory, especially in H_v-structure theory introduced in 1990. This connection appeared first in 1996 and actually several H_v-fields, the e-hyperfields, can be used as isofields or genofields so as, in such way they should cover additional properties and satisfy more restrictions. The hyperstructure theory obtained a lot of results and applications in mathematics as well as in other applied sciences. Several large classes of hyperstructures as the P-hyperfields, the ∂-hyperfields, and other classes can be used in Lie-Santilli's theory especially when multivalued problems appeared, either in finite or in infinite case. We review some of these topics and we present the Representations Theory of Hyperstructures.

Key words: Lie-Santilli iso-theory, hyperstructures, hope, H_v-structures, P-fields, ∂-fields.
AMS Subject Classification: 20N20, 16Y99

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1. INTRODUCTION

In my paper [58]: T. Vougiouklis, *The Santilli's theory 'invasion' in hyperstructures,* Algebras, Groups and Geometries 28(1), 2011, 83-103, there is a first description on how Santilli's theories effect in hyperstructures and how new theories in Mathematics can be appeared by Santilli's pioneer research. We continue with new interesting topics in this direction.

Last decades hyperstructures have applications in mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, in the Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too. For these applications the largest class of the hyperstructures, the class *H_v-structures*, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools of this theory are the fundamental relations which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on, as well. The definition of the general hyperfield was not possible without the H_v-structures and their fundamental relations. *Hypernumbers or H_v-numbers* are called the elements of H_v-fields and they are important for the representation theory.

The hyperstructures were introduced by F.Marty in 1934 [22] when he first defined the hypergoup as a set equipped with an associative and reproductive hyperoperation. The motivating example was the quotient of a group by any, not necessarily normal, subgroup. M.Koskas in 1970 [19] was introduced the fundamental relation β^*, which it turns to be the main tool in the study of hyperstructures. T.Vougiouklis in 1990 [41] was introduced the H_v-structures, by defining the weak axioms. The motivating example of those hyperstructures is the quotient of any group by any partition. Therefore the class of H_v-structures is the largest class of hyperstructures.

Motivation for H_v-structures:

The quotient of a group with respect to an invariant subgroup is a group.
The quotient of a group with respect to any subgroup is a hypergroup.
The quotient of a group with respect to any partition is an H_v-group.

In 1996 R.M.Santilli & T.Vougiouklis [33], point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. These hyperstructures contain a unique left ant right scalar unit, which is the most

important tool in Lie-Santilli theory. In what follows we present the related hyperstructure theory mainly from the paper [58], enriched with some new results on the related hyperstructures. However one can see the books and related papers for more definitions and results on hyperstructures and related topics: [3], [4], [6], [7], [8], [9], [13], [14], [19], [20], [21], [22], [36], [38], [39], [44], [45], [46], [47], [48], [50], [53], [54], [55], [57], [61], [62], [63], [71], [73].

2. BASIC DEFINITIONS ON HOPES

In a set H is called ***hyperoperation*** (abr. ***hope***) or *multivalued operation*, any map from H×H to the power set of H. Therefore, in a hope

$$\cdot : H\times H \to \tilde{A}(H): (x,y) \to x\cdot y \grave{I} H$$

the result is subset of H, instead of element as we have in usually operations.

In a set H equipped with a hope $\cdot: H\times H \to P(H)-\{\varnothing\}$, we abbreviate by

WASS the *weak associativity*: $(xy)z \cap x(yz) \neq \varnothing$, $\forall x,y,z \in H$ and by

COW the *weak commutativity*: $xy \cap yx \neq \varnothing$, $\forall x,y \in H$.

The hyperstructure $(H,\cdot)$ is called *H_v-semigroup* if it is *WASS* and it is called ***H_v-group*** if it is reproductive H_v-semigroup, i.e. $xH=Hx=H$, $\forall x \in H$. The hyperstructure $(R,+,\cdot)$ is called *H_v-ring* if $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$, and $(\cdot)$ is *weak distributive* to $(+)$:

$$x(y+z) \cap (xy+xz) \neq \varnothing, \quad (x+y)z \cap (xz+yz) \neq \varnothing, \quad \forall x,y,z \in R.$$

An extreme class of hyperstructures is the following [40]: An H_v-structure is ***very thin*** iff all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one.

An H_v-group is called *cyclic* [35], if there is an element, called *generator*, which the powers have union the underline set. The minimal power with the above property is called *period* of the generator. Moreover if there exist an element and a special power, the minimum one, is the underline set, then the H_v-group is called *single-power cyclic*.

The main tool to study all hyperstructures are the fundamental relations β^*, γ^* and ε^*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [44].

A way to find fundamental classes is given by analogous theorems to the following:

Theorem 2.1 Let $(H,\cdot)$ be an H_v-group and U be all finite products of elements of H. We define the relation β by setting $x\beta y$ iff $\{x,y\}\subset u$, $u\in U$. Then β^* is the transitive closure of β.

We present the proof for the analogous to above theorem in the case of an H_v-ring:

Theorem 2.2 Let $(R,+,\cdot)$ be H_v-ring, U the set of finite polynomials of elements of R. Define the relation γ in R as follows: $x\gamma y$ iff $\{x,y\}\subset u$ where $u\in U$. Then γ^* is the transitive closure of γ.

Proof. Let χ be transitive closure of γ, and $\chi(a)$ the class of a. First we prove that M/χ is ring.

In R/χ the sum ($\oplus$) and the product ($\otimes$) are defined in the usual manner:

$$\chi(a)\oplus\chi(b)=\{\chi(c): c\in\chi(a)+\chi(b)\},\ \gamma^*(a)\otimes\chi(b)=\{\chi(d): d\in\gamma^*(a)\cdot\chi(b)\},\ \forall a,b\in R.$$

Take $a'\in\chi(a)$, $b'\in\chi(b)$. Then we have

$$a'\chi a \text{ iff } \exists x_1,\ldots,x_{m+1} \text{ with } x_1=a',\ x_{m+1}=a \text{ and } u_1,\ldots,u_m\in U\text{: } \{x_i, x_{i+1}\}\subset u_i,\ i=1,\ldots,m, \text{ and}$$

$$b'\chi b \text{ iff } \exists y_1,\ldots,y_{n+1} \text{ with } y_1=b',\ y_{n+1}=b \text{ and } v_1,\ldots,v_n\in U\text{: } \{y_j,y_{j+1}\}\subset v_j,\ j=1,\ldots,n.$$

From the above we obtain

$$\{x_i, x_{i+1}\}+y_1\subset u_1+v_1,\ i=1,\ldots,m-1,\ x_{m+1}+\{y_j, y_{j+1}\}\subset u_m+v_j,\ j=1,\ldots,n.$$

The $u_i+v_1=t_i$, $i=1,\ldots,m-1$ and $u_m+v_j=t_{m+j-1}$, $j=1,\ldots,n$, are polynomials, so $t_k\in U$, $\forall k\{1,\ldots,m+n-1\}$.

Now, pick up $z_1,\ldots,z_{m+n}$: $z_i\in x_i+y_1$, $i=1,\ldots,n$ and $z_{m+j}\in x_{m+1}+y_{j+1}$, $j=1,\ldots,n$, thus using the above relations we obtain $\{z_k,z_{k+1}\}\subset t_k$, $k=1,\ldots,m+n-1$.

So every element $z_1\in x_1+y_1=a'+b'$ is χ equivalent to $z_{m+n}\in x_{m+1}+y_{n+1}=a+b$. Therefore, $\chi(a)\oplus\chi(b)$ is singleton and we write $\chi(a)\oplus\chi(b)=\chi(c)$, $\forall c\in\chi(a)+\chi(b)$.

In a similar way we prove that $\chi(a)\otimes\chi(b)=\chi(d)$, $\forall d\in\chi(a)\cdot\chi(b)$.

The WASS and the weak distributivity on R guarantee that the associativity and the distributivity are valid for the quotient R/γ^*. Therefore R/γ^* is a ring.

Now let σ be equivalence relation in R such that R/σ is a ring, denote $\sigma(a)$ the class of a. Then $\sigma(a)\oplus\sigma(b)$ and $\sigma(a)\otimes\sigma(b)$ are singletons for all $a,b\in R$, i.e.

$$\sigma(a)\oplus\sigma(b)=\sigma(c),\ \forall c\in\sigma(a)+\sigma(b),\ \sigma(a)\otimes\sigma(b)=\sigma(d),\ \forall d\in\sigma(a)\cdot\sigma(b).$$

Thus we can write, $\forall a,b\in R$ and $A\subset \sigma(a)$, $B\subset\sigma(b)$,

$$\sigma(a)\oplus\sigma(b)=\sigma(a+b)=\sigma(A+B),\ \ \sigma(a)\otimes\sigma(b)=\sigma(ab)=\sigma(A\cdot B).$$

By induction, we extend these relations on finite sums and products. Thus, for every $u\in U$, we have the relation $\sigma(x)=\sigma(u)$, $\forall x\in u$. Consequently, $x\in\gamma(a)$ implies $x\in\sigma(a)$, $\forall x\in R$.
But σ is transitively closed, so we obtain: $x\in\chi(x)$ implies $x\in\sigma(a)$.
That means that χ is the smallest equivalence relation in R such that R/χ is a ring, i.e. $\chi=\gamma^*$. ■

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions. Thus, to define the H_v-field the γ^* is used [41]: A H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if R/γ^* is a field. In the sequence the *H_v-vector space* is defined.

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f\in Aut(H,*) \text{ such that } xy\subset f(x*y),\ \forall x,y\in H.$$

Then we write $\cdot\leq *$ and we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called *H_b-structure*.

The little Theorem 2.3 Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

The definition of H_v-field introduced a new class of hyperstructures [55]:

Definition 2.4 The H_v-semigroup $(H,\cdot)$ is called ***h/v-group*** if the quotient H/β^* is a group.

The h/v-groups are a generalization of the H_v-groups since in h/v-groups the reproductivity is not necessarily valid. Sometimes a kind of *reproductivity of classes* is valid. This leads the quotient to be reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc, are defined. The motivation to this concept is the following: Let ω^* be the kernel of the canonical map from R to R/γ^*; then we call *reproductive H_v-field* any H_v-field $(R,+,\cdot)$ if the following axiom is valid:

$$x(R-\omega^*) = (R-\omega^*)x = R-\omega^*,\ \ \forall x\in R-\omega^*.$$

The problem of enumeration and classification of hyperstructures, was started from the beginning [35], however it becomes complicate in H_v-structures because we have very great numbers. The partial order in H_v-

structures and the Little Theorem, transfers and restrict the problem in finding the minimal, *up to isomorphisms*, H_v-structures. In this direction we have results by Bayon & Lygeros [2]:

Let H={a,b} a set of two elements. There are 20 H_v-groups, up to isomorphism,

Suppose in H={e,a,b}, a hope is defined and there exists a scalar unit, then, there are 13 *minimal* H_v-groups. The number of all H_v-groups with three elements, up to isomorphism, which have a scalar unit, is 292.

In a set with three elements there are, exactly 6.494 minimal H_v-groups. 137 are abelians and the 6.357 are non-abelians; the 6.152 are cyclic and the 342 are not cyclic.

The number of H_v-groups with three elements, up to isomorphism, is 1.026.462. There are 7.926 abelians and 1.018.536 non-abelians; the 1.013.598 are cyclic and the 12.864 are not. The 16 of them are very thin.

3. ENLARGING AND REDUCING HYPERSTRUCTURES

In [49] the 'enlarged' hyperstructures were examined in the sense that an extra element, outside the underlying set, appears in one result. In both directions, enlargement or reduction, most useful in representation theory, are those H_v-structures with the same fundamental structure: Suppose we have a structure and one element, outside of the structure, then we can attach this element in order to have a hyperstructure which becomes h/v-structure. Moreover we have the opposite problem: How one can remove at least one element of an H_v-structure or a classical structure?

The Attach Construction 3.1 Let $(H,\cdot)$ be an H_V-semigroup and $v\notin H$. We extend the $(\cdot)$ into the $\underline{H}=H\cup\{v\}$ as follows: $x\cdot v=v\cdot x=v$, $\forall x\in H$, and $v\cdot v=H$.

The $(\underline{H},\cdot)$ is a h/v-group where $(\underline{H},\cdot)/\beta^*\cong Z_2$ and v is a single element.

We call the hyperstructure $(\underline{H},\cdot)$ the *attach h/v-group* of $(H,\cdot)$.

In the representation theory of H_v-groups by H_v-matrices one needs H_v-rings or H_v-fields which have non-degenerate fundamental structures in addition with only few of hypersums and hyperproducts have cardinals greater than 1.

Theorem 3.2 Let $(G,\cdot)$ be semigroup and $v\notin G$ be an element appearing in a product ab, where $a,b\in G$, thus the result becomes a hyperproduct $a\otimes b=\{ab,v\}$. Then the minimal hope $(\otimes)$ extended in $G'=G\cup\{v\}$ such that $(\otimes)$ contains $(\cdot)$ in

the restriction on G, and such that $(G',\otimes)$ is a minimal H_V-semigroup which has fundamental structure isomorphic to $(G,\cdot)$, is defined as follows:

$$a\otimes b=\{ab,v\},\quad x\otimes y=xy,\quad \forall(x,y)\in G^2-\{(a,b)\}$$

$$v\otimes v=abab,\quad x\otimes v=xab \text{ and } v\otimes x=abx,\quad \forall x\in G.$$

Therefore, $(G',\otimes)$ is a very thin H_v-semigroup.

If $(G,\cdot)$ is commutative then the $(G',\otimes)$ becomes strongly commutative.

Definitions 3.3 Let $(H,\cdot)$ be a hypergroupoid.

We say that *remove* $h\in H$, if simply consider the restriction of $(\cdot)$ on $H-\{h\}$.

We say that $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h, whenever it appears, by $\underline{h}$.

We say that $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of $x\in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

The *uniting elements* method was introduced by Corsini & Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

The ***uniting elements*** method is the following: Let G be algebraic structure and let d be a property, which is not valid and it is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is an H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β^*, a stricter structure $(G/d)/\beta^*$ for which the property d is valid, is obtained.

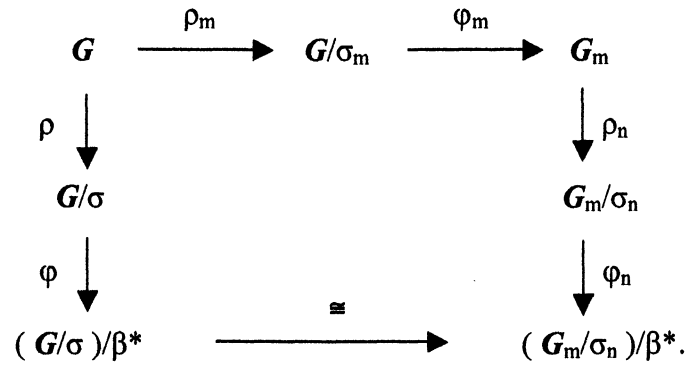
An interesting application of the uniting elements is when more than one properties are desired. The reason for this is that some of the properties lead straighter to the classes than others. So, it is better to apply the straightforward classes followed by the more complicated ones. The commutativity and reproductivity are easy applicable properties. One can do this because the following Theorem is valid.

Theorem 3.4 [44] Let $(G,\cdot)$ be a groupoid, and $F=\{f_1,\ldots, f_m, f_{m+1},\ldots, f_{m+n}\}$ be a system of equations on G consisting of two subsystems $F_m=\{f_1,\ldots,f_m\}$ and $F_n=\{f_{m+1},\ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m resp., and let σ_n be the

equivalence relation defined using the induced equations of F_n on the grupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

i.e. the following diagram is commutative



Where all the maps $\rho, \varphi, \rho_m, \varphi_m, \rho_n, \varphi_n,$ are the canonicals.

From the above it is clear that the fundamental structure is very important, mainly if it is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures. Examples how to obtain properties as associativity, commutativity, reproductivity can be found in [44].

We conclude studying the small h/v-fields.

Construction 3.5 On the ring $(Z_4,+,\cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,

(a) very thin minimal,
(b) COW (non-commutative),
(c) they have 0 and 1, scalars.

We have the isomorphic cases: $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$. The fundamental classes are $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have $(Z_4,+,\otimes)/\gamma^* \cong (Z_2,+,\cdot)$.

Thus it is isomorphic to $(Z_2\times Z_2,+)$. In this H_v-group there is only one unit and every element has a unique double inverse.

We can also define the analogous cases for the rings $(Z_6,+,\cdot)$, $(Z_9,+,\cdot)$, and $(Z_{10},+,\cdot)$.

4. THE ∂-HOPES, THE P-HOPES

In [56] a hope, in a groupoid with a map on it, called *theta* ∂, is introduced.

Definitions 4.1 Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be map. We define a hope (∂), called ***theta*** and we write ***∂-hope***, on G as follows

$$x\partial y=\{f(x)\cdot y,\ x\cdot f(y)\},\ \forall x,y\in G.\ (\text{resp. } x\partial y=(f(x)\cdot y)\cup(x\cdot f(y),\ \forall x,y\in G)$$

If $(\cdot)$ is commutative then (∂) is commutative. If $(\cdot)$ is *COW*, then (∂) is *COW*.

Let $(G,\cdot)$ a groupoid (resp. hypergroupoid) and $f:G\to \boldsymbol{P}(G)-\{\varnothing\}$ multivalued map. We define the (∂), on G as follows

$$x\partial y=(f(x)\cdot y)\cup(x\cdot f(y),\ \forall x,y\in G.$$

Let $(G,\cdot)$ be a groupoid, $f_i:G\to G$, $i\in I$, be a set of maps on G. The

$$f_\cup:G\to \boldsymbol{P}(G):\ f_\cup(x)=\{f_i(x)\mid i\in I\},$$

is *the union* of $f_i(x)$. We have the *union theta-hope* (∂), on G if we take $f_\cup(x)$. If we take $\underline{f}=f\cup(id)$, then we have the *b-theta-hope:* $\underline{\partial}$.

This definition can be generalized as follows:

Definition 4.2 Let H be a set equipped with n operations (or hopes) $\otimes_1,\ldots,\otimes_n$ and a map (or multivalued map) $f:H\to H$ (or $f:H\to P(H)$, resp.), then n hopes $\partial_1,\ldots,\partial_n$ on H can be defined, called *theta-hopes* by putting

$$x\partial_i y=\{f(x)\otimes_i y,\ x\otimes_i f(y)\},\ \forall x,y\in H \text{ and } i\in\{1,2,\ldots,n\}$$

or, in case where $\otimes_i$ is hope or f is multivalued map, we have

$$x\partial_i y=(f(x)\otimes_i y)\cup(x\otimes_i f(y)),\ \forall x,y\in H \text{ and } i\in\{1,2,\ldots,n\}$$

If $\otimes_i$ is associative then ∂_i is WASS.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for two functions s(x), t(x), we have $s\partial t=\{s't, st'\}$ where (′) denotes the derivative.

Example 4.3 Take the derivative and all first degree polynomials $g_i(x)=a_ix+b_i$. We have

$$g_1\partial g_2=\{a_1a_2x+a_1b_2,\ a_1a_2x+b_1a_2\},$$

so this is a hope in the set of the first degree polynomials. All x+c, c be constant, are units.

Properties 4.4 If $(G,\cdot)$ is a semigroup then: For every f, the hope (∂) is *WASS*, and the b-theta-operation $(\underline{\partial})$ is *WASS*. If f is homomorphism and projection, then (∂) is associative.

Reproductivity. If $(\cdot)$ is reproductive then (∂) is also reproductive.

Commutativity. If $(\cdot)$ is commutative then (∂) is commutative. If f is into the centre of G, then (∂) is commutative. If $(\cdot)$ is *COW* then, (∂) is *COW*.

Unit elements. u is unit if f(u)=e, e unit of $(G,\cdot)$. The elements of the kernel of f, are ∂-units.

Inverse elements. Let $(G,\cdot)$ monoid with unit e, u unit in (G,∂), so f(u)=e. The $x'=(f(x))^{-1}u$ and $x'=u(f(x))^{-1}$, are right and left inverses, resp. We have two-sided inverses iff f(x)u=uf(x).

Proposition. Let $(G,\cdot)$ be a group then, $\forall f:G\to G$, the (G,∂) is an H_v-group.

A connection of merge with ∂-hope: Take f: $f(\underline{h})=h$ and f(x)=x in rest cases.

One can define theta-hopes on more complicate structures.

Definition 4.5 Let $(R,+,\cdot)$ be a ring and $f:R\to R$, $g:R\to R$ be two maps. We define two hopes (∂_+) and $(\partial\cdot)$, called both *theta-hopes*, on R as follows

$$x\partial_+y=\{f(x)+y,\ x+f(y)\} \text{ and } x\partial\cdot y=\{g(x)\cdot y,\ x\cdot g(y)\},\ \forall x,y\in G.$$

$(R,+,\cdot)$, where $(+)$, $(\cdot)$ satisfy H_v-ring axioms, except weak distributivity, is an *H_v-near-ring.*

Proposition 4.6 Let $(R,+,\cdot)$ ring and $f:R\to R$, $g:R\to R$ maps. The hyperstructure $(R,\partial_+,\partial\cdot)$, called *theta*, is an H_v-near-ring. Moreover $(+)$ is commutative.

An important new application, which combines hyperstructure theory and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [70]. The suggestion is the following:

Definition 4.7 In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question.

The use of the Vougiouklis & Vougiouklis bar instead of a Likert scale has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

A well known class of hyperstructures based on classical ones are defined by [35], [38]:

Definition 4.8 Let $(G,\cdot)$ be groupoid, then for every $P\subset G$, $P\neq\emptyset$, we define the following hopes called *P-hopes*: $\forall x,y\in G$

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),\quad \underline{P}_r: x\underline{P}_r y=(xy)P\cup x(yP),\quad \underline{P}_l: x\underline{P}_l y=(Px)y\cup P(xy).$$

$(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. The most usual case is if $(G,\cdot)$ is semigroup, then

$$x\underline{P}y=(xP)y\cup x(Py)=xPy$$

and $(G,\underline{P})$ is a semihypergroup

A ***generalization of P-hopes***, used in Santilli's isotheory, is the following [10], [11], [12], [66]: Let $(G,\cdot)$ be abelian group and P a subset of G with $\#P>1$. We define the hope $(\times_P)$ as follows:

$$x\times_P y = \begin{cases} x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } c\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G,\times_P)$ is abelian H_v-group.

New definitions on this topic appear [68], [69]:

Definition 4.9 Let $A=(a_{ij})$, $B=(b_{ij})\in\mathbf{M}_{m,n}$, we call (A,B) a *unitize pair* of matrices if $A^tB=I_n$, where I_n denotes the $n\times n$ unit matrix.

The following theorem can be applied in the classical theory.

Theorem. If $m<n$, then there is no unitize pair.

Proof. Suppose that $n>m$ and that $A^tB=(c_{ij})$, $c_{ij}=\sum_{k=1}^{m} a_{ik}b_{kj}$. Denote by A_m the block of the matrix A such that $A_m=(a_{ij})\in M_{m,m}$, i.e. we take the matrix of the first m columns. Then we suppose that we have $(A_m)^tB_m=I_m$, therefore we must have $\det(A_m)\neq 0$. Now, since $n>m$, we can consider the homogeneous system with respect to the 'unknowns' $b_{1n},b_{2n},\ldots,b_{mn}$:

$$c_{in}=\sum_{k=1}^{m} a_{ik}b_{kn}=0 \quad \text{for } i=1,2,\ldots,m.$$

From which, since $\det(A_m)\neq 0$, we obtain $b_{1n}=b_{2n}=\ldots=b_{mn}=0$. Using this on the last equation, on the same unknowns, $c_{nn}=\sum_{k=1}^{m} a_{nk}b_{kn}=1$ we have $0=1$, absurd. ■

5. THE H_V-LIE ALGEBRAS

Definition 5.1 [44]. Let $(F,+,\cdot)$ be H_v-field, $(V,+)$ a COW H_v-group and there exists an external hope

$$\cdot : F\times V\to P(V): (a,x)\to ax$$

such that, $\forall a,b\in F$ and $\forall x,y\in V$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\quad (a+b)x\cap(ax+bx)\neq\varnothing,\quad (ab)x\cap a(bx)\neq\varnothing,$$

then V is called an ***H_v-vector space*** over F.

In the case of an H_v-ring instead of H_v-field then the ***H_v-modulo*** is defined.

In the above cases the fundamental relation ε^* is the smallest equivalence rsuch that the quotient V/ε^* is a vector space over the fundamental field F/γ^*.

The general definition of an H_v-Lie algebra is the following [52], [60], [37]:

Definition 5.2 Let $(L,+)$ be H_v-vector space on $(F,+,\cdot)$, $\varphi: F\to F/\gamma^*$, canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\varphi': L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Consider the bracket hope (commutator):

$$[\,,\,] : L\times L\to P(L): (x,y)\to [x,y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e. $\forall x,x_1,x_2,y,y_1,y_2\in L$ and $\forall\lambda_1,\lambda_2\in F$

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing,\quad [x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing$$

(L2) $[x,x]\cap\omega_L\neq\varnothing$, $\forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing$, $\forall x,y\in L$

This is a general definition thus one can use special cases in order to face problems in applied sciences. Moreover, we see how the weak properties can be defined as the above weak linearity (L1), anti-commutativity (L2) and the Jacobi identity (L3). Similarly the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc, are defined.

We can obtain results from special cases by applying ∂-hopes.

Definition 5.3 Let $\boldsymbol{L}$ be a Lie algebra, defined on an algebra $(V,+,\cdot)$ over the field $(F,+,\cdot)$ where the Lie bracket $[x,y]=xy-yx$. Consider any map $f:\boldsymbol{L}\to\boldsymbol{L}$, then the ∂-hope is defined as follows $\forall x,y\in\boldsymbol{L}$

$$x\partial y=\{f(x)y-f(y)x,\ f(x)y-yf(x),\ xf(y)-f(y)x,\ xf(y)-yf(x)\}$$

We conclude this paragraph with a definition of a hypergroupoid algebra.

Definition 5.4 Let $(G,\cdot)$ hypergroupoid. Is called *set of fundamental maps* on G, the set

$$Q=\{q:G\times G\to G:(x,y)\xrightarrow{onto} z\,|\,z\in xy\}.$$

Any subset $Q_s\subset Q$ defines a hope $(\circ_s)$ on G as follows

$$x\circ_s y=\{z\,|\,z=q(x,y)\text{ for some }q\in Q\}.$$

$\circ_s \le \cdot$ and $Q_s\subset Q_{os}$, where Q_{os} is the set of fundamental maps of $(\circ_s)$. A $Q_a\subset Q$ for which every $Q_s\subset Q_a$ has $(\circ_s)$ associative (resp. *WASS*) is called *associative* (resp. *WASS*). $(G,\cdot)$ is q-*WASS* if there exists a $q_o\in Q$ which defines an associative $(\circ)$ in G. For H_v-groups we have $Q\neq\varnothing$.

Suppose G is finite, $|G|=n$, and q-*WASS* with associative $q_o\in Q$. In K[G] of all formal linear combinations of elements of G with coefficients from K, we define an operation (+):

$$(f_1+f_2)(g)=f_1(g)+f_2(g),\qquad \forall g\in G,f_1,f_2\in K[G]$$

and a hope $(*)$, the *convolution*,

$$f_1*f_2=\{f_q:\ f_q(g)=\sum_{q(x,y)=g} f_1(x)f_2(y),\ q\in Q\}.$$

Definition 5.5 The hyperstructure $(K[G],+,*)$ is a multiplicative H_v-ring where the inclusion distributivity is valid. This H_v-ring is called *hypergroupoid H_v-algebra.*

$\forall q\in Q$, $g\in G$, we have

$$|Q|\le\prod_{(x,y)\,in\,G\times G}(|xy|),\quad 1\le|q^{-1}(g)|\le n^2-n+1\ \text{ and }\ \sum_{g\,in\,G}|q^{-1}(g)|=n^2.$$

The zero map $f(x)=0$ is a scalar element in K[G].

6. REPRESENTATIONS

Representations (we abbreviate by ***rep***) of H_v-groups, can be considered either by H_v-matrices [39], [43], [44] or by generalized permutations [42].

H_v-matrix (or *h/v-matrix*) is called a matrix with entries elements of an H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of H_v-matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$, of type m×n and n×r, respectively, is a set of m×r H_v-matrices, defined in a usual manner:

$$\mathbf{A}\cdot\mathbf{B} = (a_{ij})\cdot(b_{ij}) = \{\mathbf{C}=(c_{ij}) \mid c_{ij}\in\oplus\Sigma a_{ik}\cdot b_{kj}\},$$

where $(\oplus)$ is the *n-ary circle hope* on the hyperaddition: the sum of products of elements of the H_v-ring is considered to be the union of the sets obtained with all possible parentheses. However, in the case of 2×2 H_v-matrices the 2-ary circle hope which coincides with the hyperaddition in the H_v-ring. Notice that the hyperproduct of H_v-matrices does not nessesarily satisfy *WASS*.

The rep problem by H_v-matrices is the following:

Definition 6.1 Let $(H,\cdot)$ be H_v-group, $(R,+,\cdot)$ be H_v-ring and $\mathbf{M}_R=\{(a_{ij}) \mid a_{ij}\in R\}$, then any

$$\mathbf{T}:H\to\mathbf{M}_R:\ h\to\mathbf{T}(h) \text{ with } \mathbf{T}(h_1h_2)\cap\mathbf{T}(h_1)\mathbf{T}(h_2)\neq\varnothing,\ \forall h_1,h_2\in H,$$

is called ***H_v-matrix rep.*** If $T(h_1h_2)\subset T(h_1)T(h_2)$, then T is an *inclusion rep,* if $T(h_1h_2)=T(h_1)T(h_2)$, then T is *a good rep* an induced rep T* for the hypergroup algebra is obtained. If T is one to one and good then it is a *faithful rep.*

In reps of H_v-groups by H_v-matrices, there are two difficulties: To find an H_v-ring and an appropriate set of H_v-matrices. The problem of reps is very complicated mainly because the cardinality of the product of two H_v-matrices is normally very big. The problem can be simplified in several special cases such as the following:

(a) The H_v-matrices are on H_v-rings with 0 and 1 and if these are scalars.Thus the e-hyperstructures are interesting in the rep theory.
(b) The H_v-matrices are over *very thin* H_v-rings.
(c) The case of 2×2 H_v-matrices, since the 2-ary circle hope coincides with the hyperaddition in H_v-rings.
(d) The H_v-rings in which the strong associativity in hyperaddition is valid.
(e) The case of H_v-rings which contains singles, then these act as absorbings.

The main theorem of reps on H_v-structures [44], is the following:

Theorem 6.2 A necessary condition in order to have an inclusion rep T of an H_v-group $(H, \cdot)$ by n×n H_v-matrices over the H_v-ring $(R,+, \cdot)$ is the following: For all $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{ A = (a'_{ij}) \mid a'_{ij} \in \gamma^* (a_{ij}),\ i,j\in\{1,...,n\} \}$$

Thus, every inclusion rep $T:H\to M_R$: $a\mapsto T(a)=(a_{ij})$ induces a homomorphic rep T* of H/β^* over R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where the element $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. Then T* is called *fundamental induced rep* of T.

Denote $tr_\varphi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T : H \to R/\gamma^* : x \mapsto X_T(x) = tr_\varphi(T(x)) = trT^*(x)$$

is called *fundamental character*. There are several types of traces.

Several constructions are used to obtain appropriate H_v-rings and H_v-fields:

(i) Let $(H,\cdot)$ be H_v-group, then for every $(\oplus)$ such that $x\oplus y\supset\{x,y\}$, $\forall x,y\in H$, the $(H,\oplus,\cdot)$ is an H_v-ring.These H_v-rings are called *associated to* $(H,\cdot)$ H_v-rings. In reps theory of hypergroups, in sense of Marty, there are three associated hyperrings $(H,\oplus,\cdot)$ to $(H,\cdot)$. The $(\oplus)$ is defined respectively, $\forall x,y\in H$, as follows:

type a: $x\oplus y=\{x,y\}$, type b: $x\oplus y=\beta^*(x)\cup\beta^*(y)$, type c: $x\oplus y=H$.

In all types, the strong associativity and strong or inclusion distributivity, is valid.

(ii) Let $(H,+)$ be H_v-group, then for all hopes $(\otimes)$ such that $x\otimes y\supset\{x,y\}$, $\forall x,y\in H$, the $(H,+,\otimes)$ is an H_v-ring.
A variation of this is: Let $(H,+)$ be H_v-group with a scalar 0. Then $\forall(\otimes)$: $x\otimes y\supset\{x,y\}$, $\forall x,y\in H-\{0\}$, $x\otimes 0=0\otimes x=0$, $\forall x\in H$, the $(H,+,\otimes)$ is an H_v-ring.

(iii) Let $(H,\cdot)$ be H_v-group. Take a $0\notin H$ and set $H'=H\cup\{0\}$. Define a hope $(+)$ by: $0+0=0$, $0+x=H=x+0$, $x+y=0$, $\forall x,y\in H$, and extend $(\cdot)$ in H' by $0\cdot 0=0,0\cdot x=x\cdot 0=0,\forall x,y\in H$. Then $(H',+,\cdot)$ is reproductive H_v-field with $H'/\gamma^*\cong Z_2$ where 0 is absorbing and single.

Some open problems arising on the topic of rep theory of hypergroups, are:

Open Problems.

a. Find standard H_v-rings or H_v-fields to represent all H_v-groups by H_v-matrices.

b. Find reps by H_v-matrices over standard finite H_v-rings analogous to Z_n.
c. On matrices find a generalization of the multiplication of matrices to use in H_v-rep theory.
d. Find reps of special classes of hypergroups and reduce these to minimal dimensions.

Hopes on any type of ordinary matrices can be defined [72], [73], [15] they are called *helix hopes*.

Definition 6.3 Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$ be matrix and $s,t\in N$, with $1\le s\le m$, $1\le t\le n$. The *helix-projection* is a map $\underline{st}:\mathbf{M}_{m\times n}\to\mathbf{M}_{s\times t}:A\to A\underline{st}=(\underline{a}_{ij})$, where $A\underline{st}$ has entries

$$\underline{a}_{ij}=\{\, a_{i+\kappa s,j+\lambda t}\mid 1\le i\le s,\ 1\le j\le t \text{ and } \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n\,\}$$

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ be matrices and $s=\min(m,u)$, $t=\min(n,v)$. We define a hyper-addition, called *helix-sum*, by

$$\oplus:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{s\times t}):(A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset\mathbf{M}_{s\times t}$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij})\mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ and $s=\min(n,u)$. Define the *helix-product*, by

$$\otimes:\mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{m\times v}):(A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset\mathbf{M}_{m\times v}$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj})\mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

The helix-sum is commutative, WASS, not associative. The helix-product is WASS, not associative and not distributive to the helix-addition.

Using several classes of H_v-structures one can face several representations. Some of those classes are as follows [68], [69], [16], [17]:

Definition 6.4 Let $M=M_{m\times n}$, the set of m×n matrices on R and $P=\{P_i:i\in I\}\subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on M as follows

$$\underline{P}:M\times M\to P(M):(A,B)\ A\underline{P}B=\{AP^t{}_iB:i\in I\}\subseteq M$$

where P^t denotes the transpose of P.

$\underline{P}$ is bilinear Rees' like operation where, instead of one sandwich matrix, a set is used. $\underline{P}$ is strong associative and inclusion distributive to addition:

$$A\underline{P}(B+C)\subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C\in M$$

is valid. So $(M,+,\underline{P})$ defines a multiplicative hyperring on non-square matrices.
Definition 6.5 Let $\mathbf{M}=\mathbf{M}_{m\times n}$ be module of m×n matrices on R and take the sets

$$\mathbf{S}=\{s_k:k\in K\}\subseteq R,\ \mathbf{Q}=\{Q_i:j\in J\}\subseteq \mathbf{M},\ \mathbf{P}=\{P_i:i\in I\}\subseteq \mathbf{M}.$$

Define three hopes as follows

$$\underline{S}: R\times \boldsymbol{M}\to \boldsymbol{P}(\boldsymbol{M}): (r,A)\to r\underline{S}A = \{(rs_k)A: k\in K\}\subseteq \boldsymbol{M}$$

$$\underline{Q}_+: \boldsymbol{M}\times\boldsymbol{M}\to \boldsymbol{P}(\boldsymbol{M}): (A,B)\to A\underline{Q}_+B = \{A+Q_j+B: j\in J\}\subseteq \boldsymbol{M}$$

$$\underline{P}: \boldsymbol{M}\times\boldsymbol{M}\to \boldsymbol{P}(\boldsymbol{M}): (A,B)\to A\underline{P}B = \{AP^t{}_iB: i\in I\}\subseteq \boldsymbol{M}$$

Then $(\boldsymbol{M},\underline{S},\underline{Q}_+,\underline{P})$ is a hyperalgebra on $\boldsymbol{R}$ called *general matrix P-hyperalgebra.*

We present the rep problem is by *Generalized Permutations* (write ***gp***) [42].

Definitions 6.6 Let X be a set, then a map $f:X\to P(X)-\{\varnothing\}$, is a *gp of X* if the reproduction axiom is valid $\cup_{x\in X}f(x)=f(X)=X$. Denote by M_X the set of all gps on X. For an H_v-group $(X,\cdot)$ and $a\in X$, the gp f_a defined by $f_a(x)=ax$ is *inner gp.* *Arrow* of f is any $(x,y)\in X^2$ with $y\in f(x)$. $f_2\in M_X$ *contains* $f_1\in M_X$ or f_1 is a *sub-gp* of f_2, if $f_1(x)\subset f_2(x)$, $\forall x\in X$, then we write $f_1\subset f_2$. If, moreover, $f_1\neq f_2$, then f_1 is a proper sub-gp of f_2. A $f\in M_X$ is called *minimal* if it has no proper sub-gp. Denote $\underline{M}_X$ the set of all minimal gps of M_X. The gp t with $t(x)=X$, $\forall x\in X$, is called *universal* and contains all elements of M_X. The *converse* of a gp f is the gp $\underline{f}$ defined by $\underline{f}(x)=\{z\in X: f(z)\ni x\}$, thus $\underline{f}$ is obtained by reversing arrows. We call *associated* to $f\in M_X$ the gp $f\circ\underline{f}$, where $(\circ)$ is the map composition.

The *Union* $f=\cup_{i\in I} f_i$ of a family of gps $\{f_i: i\in I\}$, is $f(x)=\cup_{i\in I} f_i(x)$, $\forall x\in X$.

For finite X, we reach a minimal gp, by the *deleting arrows* method.

Theorem 6.7 Let $f\in M_X$, then $f\in\underline{M}_X$ if and only if, the following condition is valid: if $a\neq b$ and $f(a)\cap f(b)\neq\varnothing$, then $f(a)=f(b)$ and $f(a)$ is a singleton.

Corollary. If $f\in\underline{M}_X$ then $\underline{f}\in\underline{M}_X$.

Explicit description of

$$\underline{M}_X: (f\circ\underline{f})(x)=f\{u:f(u)\ni x\}=\cup_{f(u)\ni x}f(u),\ \forall x\in X.$$

Thus $(f\circ\underline{f})(x)=\{y:\exists u\in X,\{x,y\}\subset f(u)\}$. So, if I is the identity, then $I\subset f\circ\underline{f}$, $\forall f\in M_X$.

There is a direct relation of β^* and the associated gp $f\circ\underline{f}$. We see this relation, for finite X, in the following theorem:

Theorem 6.8 If $f\in\underline{M}_X$ then $(f\circ\underline{f})(x)=\{y\in X: f(y)=f(x)\}$.

To face the rep problem by the gp notice that the usual map composition is not multivalued. Thus the main *open problem* is to find standard hope on M_X.

7. THE SANTILLI'S e-CONSTRUCTIONS

Last decades the H_v-structures have applications in other branches of mathematics and in other sciences, as well. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [1],[11],[33],[34],[51],[59].

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1999, and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology.

Definition 7.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 7.2 A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ a hope, is called *e-hyperfield* if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and $\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 7.3 *The Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes (Ä) as follows:

$$x\ddot{A}y=\{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\},\text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$g_1, g_2, \ldots$ are not necessarily the same for each pair (x,y). (G,Ä) is an H_v-group, it is an H_b-group which contains the (G,·). (G,Ä) is an e-hypergroup. Moreover, if for each x,y such that xy=e, so we have xÄy=xy, then (G,Ä) becomes a strong e-hypergroup.

The proof is immediate since for both cases we enlarge the results of the group by putting elements from the set G and applying the Little Theorem. Moreover one can easily see that the unit e is a unique scalar element and for each x in G, there exists a unique inverse x^{-1}, such that $1 \in x \cdot x^{-1} \cap x^{-1} \cdot x$. Finally if the last condition is valid then we have $1 = x \cdot x^1 = x^{-1} \cdot x$, so the hyperstructure (G,Ä) is a strong e-hypergroup.

Remark that the main e-construction gives an extremely large class of e-hopes. These e-hopes can be used in the several more complicate hyperstructures to obtain the appropriate ones. The most useful are the ones where only few products are enlarged and, even more, the extra elements are one or two. This means that we have analogous cases to the rep theory.

The main e-construction gives an extremely large number of e-hopes.

Example 7.4 Consider the quaternion group $\boldsymbol{Q}=\{1,-1, i,-i, j,-j, k,-k\}$ with defining relations $i^2=j^2=-1$, ij=-ji=k. Denoting $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$ we may define a very arge number (∗) hopes by enlarging only few products. For example, (-1)∗k=$\underline{k}$, k∗i=$\underline{j}$ and i∗j=$\underline{k}$. Then the hyperstructure (Q,∗) is a strong e-hypergroup.

8. THE SANTILLI'S ISO-THEORY

According to Santilli's iso-theory [27], [28], [29], [30], [32], [64], [66], [69], [18] on a field F=(F,+,·), *a general isofield* $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where a∈F, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1}, \ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}} \qquad \text{(i)}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of Fé,

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\times\hat{1} = \hat{a}\,, \ \forall\hat{a}\in\hat{\boldsymbol{F}} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory into the hyperstructure case we generalize only the new multiplication $\hat{\times}$ from (i), by replacing with a hope including the old one. We introduce two general constructions on this direction as follows:

Construction 8.1 *General enlargement.* On a field $\boldsymbol{F}=(\boldsymbol{F},+,\cdot)$ and on the isofield $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ we replace in the results of the iso-product

$$\hat{a}\,\hat{\times}\,\hat{b}= \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{\boldsymbol{F}}$, containing $\hat{T}$, for all $\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{\boldsymbol{F}}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore the new iso-hope is

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{H}_{ab}\times\hat{b}= \hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}} \qquad \text{(iii)}$$

$\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ becomes *isoH$_v$-field.* The elements of $\boldsymbol{F}$ are called *isoH$_v$-numbers* or *isonumbers.*

Remarks 8.2 More important hopes, of the above construction, are the ones where only for few ordered pairs $(\hat{a},\hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few, preferable exactly one. Thus, this special case is if there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\,\hat{\times}\,\hat{b}= \hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b}, \quad \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}}$$

and the rest are ordinary results, then we have a hyperstructure called *very thin isoH$_v$-field.*

The assumption that $\hat{H}_{ab}=\{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, with that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH$_v$-field has one scalar absorbing $\hat{0}$, one scalar $\hat{1}$, and $\forall\hat{a}\in\hat{\boldsymbol{F}}$, has one inverse.

Construction 8.3 *The P-hope.* Consider an isofield $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ with $\hat{a}=a\times\hat{1}$, the isonumbers, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional unit 0, and $\hat{\times}$ is the iso-multiplication

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}}$$

Take a set $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, with $\hat{p}_1,\ldots,\hat{p}_s\in\hat{\boldsymbol{F}}-\{\hat{0},\hat{1}\}$, define the ***isoP-H$_v$-field***, $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times}_P)$, where the hope $\hat{\times}_P$ as follows:

$$\begin{cases} \hat{a}\times\hat{P}^{\wedge}\times\hat{b} = \{\hat{a}\times\hat{h}^{\wedge}\times\hat{b}\,|\,\hat{h}^{\wedge}\in\hat{P}^{\wedge}\} \text{ if } \hat{a}\neq\hat{1} \text{ and } \hat{b}\neq\hat{1} \\ \end{cases}$$

$$\hat{a} \otimes_P \hat{b} := \begin{cases} \hat{a} \times \hat{T}^{\wedge} \times \hat{b} & \text{if } \hat{a} = \hat{1} \text{ or } \hat{b} = \hat{1} \end{cases} \quad \text{(iv)}$$

The elements of $\hat{F}$ are called *isoP-H_v-numbers*.

Remark. If $\hat{P}=\{\hat{T},\hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. The inverses in isoP-H_v-fields, are not necessarily unique.

Example 8.4 Non degenerate example on the above constructions on the small finite field:

In order to define a generalized P-hope on $\hat{Z}_7=\hat{Z}_7(\hat{a},\hat{+},\otimes)$, where we take $\hat{P}=\{\hat{1},\hat{5}\}$, the weak associative multiplicative hope is described by the table:

$\otimes$	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$
$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$	$\hat{0}$
$\hat{1}$	$\hat{0}$	$\hat{1}$	$\hat{2}$	$\hat{3}$	$\hat{4}$	$\hat{5}$	$\hat{6}$
$\hat{2}$	$\hat{0}$	$\hat{2}$	$\hat{4},\hat{6}$	$\hat{6},\hat{2}$	$\hat{1},\hat{5}$	$\hat{3},\hat{1}$	$\hat{5},\hat{3}$
$\hat{3}$	$\hat{0}$	$\hat{3}$	$\hat{6},\hat{2}$	$\hat{2},\hat{3}$	$\hat{5},\hat{4}$	$\hat{1},\hat{5}$	$\hat{4},\hat{6}$
$\hat{4}$	$\hat{0}$	$\hat{4}$	$\hat{1},\hat{5}$	$\hat{5},\hat{4}$	$\hat{2},\hat{3}$	$\hat{6},\hat{2}$	$\hat{3},\hat{1}$
$\hat{5}$	$\hat{0}$	$\hat{5}$	$\hat{3},\hat{1}$	$\hat{1},\hat{5}$	$\hat{6},\hat{2}$	$\hat{4},\hat{6}$	$\hat{2},\hat{3}$
$\hat{6}$	$\hat{0}$	$\hat{6}$	$\hat{5},\hat{3}$	$\hat{4},\hat{6}$	$\hat{3},\hat{1}$	$\hat{2},\hat{3}$	$\hat{1},\hat{5}$

The hyperstructure $\hat{Z}_7=Z_7(\hat{a},\hat{+},\otimes)$, is commutative and associative on the multiplication hope.

9. THE LIE-SANTILLI'S ADMISSIBILITY ON NON-SQUARE MATRICES

Another important new field in hypermathematics comes straightforward from Santilli's Admissibility. We can transfer Santilli's theory in admissibility for representations in two ways: using either, the ordinary matrices and a hope on them, or using hypermatrices and ordinary operations on them [24], [25], [26], [31], [32], [34], [65], [68], [23].

The general definition is the following:

Definition 9.1 Let $\boldsymbol{L}$ be H_v-vector space over the H_v-field $(\boldsymbol{F},+,\cdot)$, $\varphi:\boldsymbol{F}\rightarrow\boldsymbol{F}/\gamma^*$, the canonical map and $\omega_F=\{x\in\boldsymbol{F}:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $\boldsymbol{F}/\gamma^*$. Let ω_L be the core of the canonical map $\varphi':\boldsymbol{L}\rightarrow\boldsymbol{L}/\varepsilon^*$

and denote by the same symbol 0 the zero of L/ε^*. Take two subsets $R,S\subseteq L$ then a ***Lie-Santilli admissible hyperalgebra*** is obtained by taking the Lie bracket, which is a hope:

$$[\,,\,]_{RS}: L\times L\to P(L): [x,y]_{RS}= xRy-ySx= \{xry-ysx \mid r\in R,\ s\in S\}$$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) When only S is considered, then $[x,y]_S= xy-ySx= \{xy-ysx \mid s\in S\}$

(b) When only R is considered, then $[x,y]_R= xRy-yx= \{xry-yx \mid r\in R\}$

(c) When $R=\{r_1,r_2\}$ and $S=\{s_1,s_2\}$ then

$$[x,y]_{RS}= xRy-ySx= \{xr_1y-ys_1x,\ xr_1y-ys_2x,\ xr_2y-ys_1x,\ xr_2y-ys_2x\}.$$

(d) When $S=\{s_1,s_2\}$ then $[x,y]_S= xy-ySx= \{xy-ys_1x,\ xy-ys_2x\}$.

(e) When $R=\{r_1,r_2\}$ then $[x,y]_R= xRy-yx= \{xr_1y-yx,\ xr_2y-yx\}$.

(f) We have one case which is 'like' P-hope for any subset $S\subseteq L$:

$$[x,y]_S = \{xsy-ysx \mid s\in S\}$$

On non square matrices we can define admissibility, as well:

Construction 9. Let $(L=M_{m,n},+)$ be H_v-vector space of m×n hyper-matrices on the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the field F/γ^*. Similarly, let ω_L be the core of $\varphi':L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Take any two subsets $R,S\subseteq L$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hope:

$$[\,,\,]_{RS}: L\times L\to P(L): [x,y]_{RS}=xR^ty-yS^tx.$$

Notice that $[x,y]_{RS}=xR^ty-yS^tx=\{xr^ty-ys^tx \mid r\in R \text{ and } s\in S\}$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) $R=\{e\}$ then $[x,y]_{RS} = xy-yS^tx = \{xy-ys^tx \mid s\in S\}$

(b) $S=\{e\}$ then $[x,y]_{RS} = xR^ty-yx = \{xr^ty-yx \mid r\in R\}$

(c) $R=\{r_1,r_2\}$ and $S=\{s_1,s_2\}$ then

$$[x,y]_{RS}=xR^ty-yS^tx=\{xr_1{}^ty-ys_1{}^tx, xr_1{}^ty-ys_2{}^tx, xr_2{}^ty-ys_1{}^tx, xr_2{}^ty-ys_2{}^tx\}$$

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HYPERNUMBERS, FINITE HYPER-FIELDS[1]

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Abstract

Last decades hyperstructures have applications in mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, in the Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too. For these applications the largest class of the hyperstructures, the class *H_v-structures*, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools of this theory are the fundamental relations which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on, as well. The definition of the general hyperfield was not possible without the H_v-structures and their fundamental relations. *Hypernumbers or H_v-numbers* are called the elements of H_v-fields and they are important for the representation theory,

We present the theory of hypernumbers and we focus on small finite H_v-fields.

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Key words: *hyperstructure, H_v-structure, hope, hypernumbers, iso-numbers.*

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1. INTRODUCTION

The object of this paper is the class of hyperstructures called *H_v-structures* introduced in 1990 [36], which satisfy the *weak axioms* where the non-empty intersection replaces the equality. Some basic definitions are the following:

Algebraic hyperstructure is a set H equipped with a *hyperoperation* (abbr. *hyperoperation=**hope***) $\cdot: H\times H\to P(H)-\{\varnothing\}$. Abbreviate by *WASS* the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by *COW* the *weak commutativity*: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$. The hyperstructure $(H,\cdot)$ is an *H_v-semigroup* if it is *WASS*, it is called ***H_v-group*** if it is reproductive H_v-semigroup, i.e. $xH=Hx=H$, $\forall x\in H$.

Motivation. The quotient of a group with respect to an invariant subgroup, is a group. F. Marty from 1934, states that, the quotient of a group by a subgroup is a hypergroup. The quotient of a group by a partition (or equivalently to any equivalence) is an H_v-group.

In an H_v-semigroup the *powers* are: $h^1=\{h\}$, $h^2=h\cdot h,\ldots$, $h^n=h\circ h\circ\ldots\circ h$, where $(\circ)$ is the *n-ary circle hope*, i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An $(H,\cdot)$ is *cyclic of period s*, if there is h, *generator*, and an s, the minimum: $H=h^1\cup h^2\cup\ldots\cup h^s$. Analogously the cyclicity for the infinite period is defined [30],[40]. If there is an h and s, the minimum: $H=h^s$, then $(H,\cdot)$, is called *single-power cyclic of period s*.

In an a similar way more complicated hyperstructures are defined:
$(R,+,\cdot)$ is called ***H_v-ring*** if $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\quad \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be H_v-ring, $(M,+)$ be a COW H_v-group and there is an external hope

$$\cdot: R\times M\to P(M): (a,x)\to ax$$

such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\ (a+b)x\cap(ax+bx)\neq\varnothing,\ (ab)x\cap a(bx)\neq\varnothing,$$

then M is called an *H_v-module* over F. In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the ***H_v-vector space*** is defined.

For more definitions and applications on H_v-structures one can see in books and papers as [2],[3],[5],[6],[7],[8],[9],[10],[16],[17],[19],[20],[22],[23],[40], [41],[42],[43],[66].

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups on the same set H, the hope $(\cdot)$ is called *smaller* than the $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f\in Aut(H,*) \quad \text{such that} \quad xy\subset f(x*y),\ \forall x,y\in H.$$

Then we write $\cdot\leq^*$ and we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called *H_b-structure*.

Theorem 1.1 *(The Little Theorem).* Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.

This Theorem leads to a partial order on H_v-structures and to posets.

Let $(H,\cdot)$ be hypergroupoid. We *remove* $h\in H$, if we take the restriction of $(\cdot)$ in $H-\{h\}$. $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$. $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$.

The main tool in hyperstructures is the *fundamental relation.* M. Koscas 1970[21] defined in hypergroups the relation β and its transitive closure β^*. This relation connects hyperstructures with the classical structures and is defined in H_v-groups as well. T. Vougiouklis [14],[15],[18],[20],[21],[23],[25] introduced the γ^* and ε^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named all these relations, *fundamental.*

Definition 1.2 The ***fundamental relations*** β^*, γ^* and ε^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.

Specifying the above motivation we remark: Let $(G,\cdot)$ be a group and R be an equivalence relation (or partition) in G, then $(G/R,\cdot)$ is H_v-group, therefore we have the quotient $(G/R,\cdot)/\beta^*$ which is a group, the *fundamental* one.

The way to find the fundamental classes is given by:

Theorem 1.3 Let $(H,\cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. Define the relation β in H by: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then β^* is the transitive closure of β.

Analogous theorems are for H_v-rings, H_v-vector spaces and so on. For proofs see [35], [36], [40], [52], [55].

An element is called ***single*** if its fundamental class is singleton [40].

More general structures can be defined by using the fundamental structures. An application in this direction is the general hyperfield. There was no general definition of a hyperfield, but from 1990 [36] there is the following [40]:

Definition 1.4 An H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if R/γ^* is a field.

Since the algebras are defined on vector spaces, the analogous to Theorem 1.3, on H_v-vector spaces is the following:

Theorem 1.5 Let $(V,+)$ be an H_v-vector space over the H_v-field F. Denote by U the set of all expressions consisting of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V. We define the relation ε, in V as follows: $x\varepsilon y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the relation ε^* is the transitive closure of the relation ε.

Definition 1.6 [32],[52], [53]. Let $(L,+)$ be an H_v-vector space over the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Let ω_L be the core of the canonical map $\varphi':L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Consider the *bracket (commutator) hope*:

$$[\ ,\ ]: L\times L\to P(L): (x,y)\to [x,y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$

$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing$, $\forall x,x_1,x_2,y,y_1,y_2\in L$ and $\lambda_1,\lambda_2\in F$

(L2) $[x,x]\cap\omega_L\neq\varnothing$, $\forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing$, $\forall x,y\in L$

Definition 1.4, introduces a new class of which is the following [46], [54]:

Definition 1.7 The H_v-semigroup $(H,\cdot)$ is called ***h/v-group*** if H/β^* is a group.

The class of h/v-groups is more general than the H_v-groups since in h/v-groups the reproductivity is not valid. The *reproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes, then $x\sigma(y)=\sigma(xy)=\sigma(x)y$, $\forall x,y\in H$, because the quotient is reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc are defined.

The uniting elements method was introduced by Corsini–Vougiouklis [4] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

The *uniting elements* method is the following: Let ***G*** be algebraic structure and d, a property which is not valid. Suppose that d is described by a set of equations; then, take the partition in ***G*** for which it is put together, in the same

class, every pair of elements that causes the non-validity of the property d. The quotient by this partition $\boldsymbol{G}/d$ is an H_v-structure. Then, quotient out the H_v-structure $\boldsymbol{G}/d$ by the fundamental relation β^*, a stricter structure $(\boldsymbol{G}/d)\beta^*$ for which the property d is valid, is obtained.

It is important if more properties are desired, then we have [40]:

Theorem 1.8 Let $(\boldsymbol{R},+,\cdot)$ be ring, and $F=\{f_1,\ldots, f_m, f_{m+1},\ldots, f_{m+n}\}$be system of equations on $\boldsymbol{R}$ consisting of subsystems $F_m=\{f_1,\ldots,f_m\}$ and $F_n=\{f_{m+1},\ldots,f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the ring $\boldsymbol{R}_m= (\boldsymbol{R}/\sigma_m)/\gamma^*$. Then

$$(\boldsymbol{R}/\sigma)/\gamma^* \cong (\boldsymbol{R}_m/\sigma_n)/\gamma^*.$$

2. CHARACTERISTIC LARGE CLASSES OF HOPES

A class of H_v-structures, introduced in [37] is the following:

Definition 2.1 An H_v-structure is called ***very thin*** if all hopes are operations except one, which has all hyperproducts singletons except one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in H there exists a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which $ab=A$, with $cardA>1$, and all the other products, are singletons.

From the very thin hopes the *Attach Construction* is obtained [44]: Let $(H,\cdot)$ be an H_V-semigroup and $v\notin H$. We extend the $(\cdot)$ into $\underline{H}=H\cup\{v\}$ by:

$$x\cdot v=v\cdot x=v,\ \forall x\in H, \quad \text{and} \quad v\cdot v=H.$$

The $(\underline{H},\cdot)$ is an H_V-group, where $(\underline{H},\cdot)/\beta^*\cong Z_2$ and v is a single.

A class of H_v-structures is the following [49]:

Definition 2.2 Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be a map. We define a hope (∂), called *theta-hope,* we write *∂-hope*, on G as follows

$$x\partial y = \{f(x)\cdot y, x\cdot f(y)\} \text{ (resp. } x\partial y = (f(x)\cdot y)\cup(x\cdot f(y))),\ \forall x,y\in G.$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is *COW*, then ∂ is *COW*.

If $(G,\cdot)$ is groupoid (or hypergroupoid) and $f:G\to P(G)-\{\varnothing\}$ be multivalued map. We define the ∂-hope on G as follows:

$$x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)),\ \forall x,y\in G.$$

∂-hopes can be defined in H_v-vector spaces and H_v-Lie algebras [57],[63]:

Let $(A,+,\cdot)$ be an algebra over F. Take any map $f:A\to A$, then the *∂-hope on the Lie bracket* $[x,y]=xy-yx$, is defined as follows

$$x\partial y = \{f(x)y-f(y)x,\ f(x)y-yf(x),\ xf(y)-f(y)x,\ xf(y)-yf(x)\}.$$

then $(A,+,\partial)$ is an H_v-algebra on F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

Motivation for the ∂-hope is the map *derivative* where only the product of functions can be used.

Basic property: if $(G,\cdot)$ is semigroup then $\forall f$, the ∂-hope is *WASS*.

Examples (a) In integers $(Z,+,\cdot)$ fix $n\neq 0$, a natural number. Consider the map f such that $f(0)=n$ and $f(x)=x$, $\forall x\in Z-\{0\}$. Then $(Z,\partial_+,\partial\cdot)$, where ∂_+ and $\partial\cdot$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(Z,\partial_+,\partial\cdot)/\gamma^* \cong Z_n.$$

(b) In $(Z,+,\cdot)$ with $n\neq 0$, take f such that $f(n)=0$ and $f(x)=x$, $\forall x\in Z-\{n\}$. Then $(Z,\partial_+,\partial\cdot)$ is an H_v-ring, moreover, $(Z,\partial_+,\partial\cdot)/\gamma^* \cong Z_n$.

Special case of the above is for n=p, prime, then $(Z,\partial_+,\partial\cdot)$ is an H_v-field.

Combining the uniting elements procedure with the enlarging theory or the ∂-theory, we can obtain analogous results [54], [55].

Theorem. In the ring $(Z_n,+,\cdot)$, with n=ms we enlarge the multiplication only in the product of the special elements $0\cdot m$ by setting $0\otimes m=\{0,m\}$ and the rest results remain the same. Then

$$(Z_n,+,\otimes)/\gamma^* \cong (Z_m,+,\cdot).$$

Remark that we can enlarge other products as well, for example $2\cdot m$ by setting $2\otimes m=\{2,m+2\}$, then the result remains the same. In this case 0 and 1 are scalars.

Corollary. In the ring $(Z_n,+,\cdot)$, with n=ps where p is prime, we enlarge only the product $0\cdot p$ by $0\otimes p=\{0,p\}$ and the rest remain the same. Then $(Z_n,+,\otimes)$ is very thin H_v-field.

An important application, which combines hyperstructure theory, especially the ∂-hyperstructures, and fuzzy theory, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [64], [65]. The suggestion is the following:

Definition 2.3 In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question.

The use of the Vougiouklis & Vougiouklis bar instead of a Likert scale has several advantages during both the filling-in and the research processing [65]. The final suggested length of the bar which is used now is, according to the Golden Ratio, is 6.2cm.

An important and well known class of hyperstructures defined on classical structures are defined as follows [30],[34],[40],[53],[18]:

Definition 2.4 Let $(G,\cdot)$ be groupoid, then for every $P\subset G$, $P\neq\emptyset$, we define the following hopes called *P-hopes*: $\forall x,y\in G$

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),\quad \underline{P}_r: x\underline{P}_r y=(xy)P\cup x(yP),\quad \underline{P}_l: x\underline{P}_l y=(Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. The most usual is if $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup

A ***generalization of P-hopes***, used in Santilli's isotheory, is the following [11],[12],[13],[14]: Let $(G,\cdot)$ be abelian group and P a subset of G with $\#P>1$. We define the hope $(\times_P)$ as follows:

$$x\times_P y = \begin{cases} x\cdot P\cdot y=\{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } c\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G,\times_P)$ is abelian H_v-group.

Some hopes 'like' P-hopes are the following [48],[53],[59]:

Definition 2.5 Let $M=M_{m\times n}$ be the $m\times n$ matrices over R and $P=\{P_i: i\in I\}\subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on M as follows

$$\underline{P}: M\times M\to P(M): (A,B)\ A\underline{P}B=\{AP^t_iB: i\in I\}\subseteq M$$

where P^t denotes the transpose of P.

The hope $\underline{P}$ is a bilinear map, strong associative and inclusion distributive:

$$A\underline{P}(B+C)\subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C\in M$$

Thus, $(M,+,\underline{P})$ defines a multiplicative hyperring on non-square matrices.

Let $\mathbf{M}=\mathbf{M}_{m\times n}$ be a module of m×n matrices over R and let us take sets

$$\mathbf{S}=\{s_k:k\in K\}\subseteq R,\ \mathbf{Q}=\{Q_i:j\in J\}\subseteq \mathbf{M},\ \mathbf{P}=\{P_i:i\in I\}\subseteq \mathbf{M}.$$

Define three hopes as follows

$$\underline{S}: R\times M\to P(M): (r,A)\to r\underline{S}A = \{(rs_k)A: k\in K\}\subseteq M$$

$$\underline{Q_+}: M\times M\to P(M): (A,B)\to A\underline{Q_+}B = \{A+Q_j+B: j\in J\}\subseteq M$$

$$\underline{P}: M\times M\to P(M): (A,B)\to A\underline{P}B = \{AP^t_iB: i\in I\}\subseteq M$$

Then $(M,\underline{S},\underline{Q_+},\underline{P})$ is a hyperalgebra on R, the *general matrix P-hyperalgebra.*

New definitions on this topic appear [59], [63]:

Definition 2.6 Let $A=(a_{ij})$, $B=(b_{ij})\in\mathbf{M}_{m\times n}$, we call a pair of matrices (A,B) *unitize pair* if $A^tB=I_n$, where I_n denotes the n×n unit matrix.

The following theorem can be applied in the classical theory [63].

Theorem. If m<n, then there is no unitize pair.

Proof. Suppose that n>m and that $A^tB=(c_{ij})$, $c_{ij}=\sum_{k=1}^{m} a_{ik}b_{kj}$. Denote by A_m the block of the matrix A such that $A_m=(a_{ij})\in M_{m\times m}$, i.e. we take the matrix of the first m columns. Then we suppose that we have $(A_m)^tB_m = I_m$, therefore we must have $\det(A_m)\neq 0$. Now, since n>m, we can consider the homogeneous system with respect to the 'unknowns' $b_{1n},\ldots, b_{mn}$:

$$c_{in}=\sum_{k=1}^{m} a_{ik}b_{kn} = 0 \quad \text{for } i=1,2,\ldots,m.$$

From which, since $\det(A_m)\neq 0$, we obtain that $b_{1n}=b_{2n}=\ldots=b_{mn}=0$. Using this fact on the last equation, on the same unknowns,

$$c_{nn}=\sum_{k=1}^{m} a_{nk}b_{kn}=1$$

we have 0=1, absurd. ■

An important hope on non-square matrices is defined [67],[19],[15],[68]:

Definitions 2.7 Let $A=(a_{ij})\in M_{m\times n}$ be m×n matrix and s,t∈N naturals such that $1\le s\le m$, $1\le t\le n$. Then we define the characteristic-like map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A, the matrix $A\underline{cst}=(a_{ij})$ where $1\le i\le s$, $1\le j\le t$, called *cut-projection* of type $\underline{st}$. Thus $A\underline{cst}$ is obtained from A by cutting the lines with index greater than s, and columns with index greater than t.

Let $A=(a_{ij})\in M_{m\times n}$ and $s,t\in N$, such that $1\le s\le m$, $1\le t\le n$. We define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st}=(\underline{a}_{ij})$ with entries the sets

$$\underline{a}_{ij}=\{a_{i+\kappa s,j+\lambda t}\mid 1\le i\le s,\ 1\le j\le t \text{ and } \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n\}.$$

Thus we have the map $\underline{st}: M_{m\times n}\to M_{s\times t}: A\to A\underline{st}=(\underline{a}_{ij})$.
This multivalued map is called *helix-projection* of type $\underline{st}$. Thus $A\underline{st}$ is a set of $s\times t$-matrices $X=(x_{ij})$ such that $x_{ij}\in\underline{a}_{ij}$, $\forall i,j$. Obviously $A\underline{mn}=A$.

Let $A=(a_{ij})\in M_{m\times n}$ and $s,t\in N$, $1\le s\le m$, $1\le t\le n$. Then it is clear that we can apply the helix-projection first on the columns and then on the rows or the opposite and the result is the same: $(A\underline{sn})\underline{st}=(A\underline{mt})\underline{st}=A\underline{st}$.

If $A\underline{st}$ is not set but a matrix then is a *cut-helix* of type $s\times t$. So $A\underline{cst}=A\underline{st}$.

Definitions 2.8 (a) Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$ with $s=\min(m,u)$, $t=\min(n,u)$. We define a hope, called *helix-addition* or ***helix-sum***, as follows:

$$\oplus: M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}): (A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij})\mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}$.

(b) Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$ be matrices and $s=\min(n,u)$. We define a hope, called *helix-multiplication* or ***helix-product***, as follows:

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}): (A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj})\mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}$.

Remark. In the set $M_{m\times n}$ the addition of matrices is an ordinary operation, thus we define the hyperproduct. The definition of the Lie-bracket is immediate, therefore the *helix-Lie Algebra* is defined [57], [68], as well.

3. H_v-MATRICES AND REPRESENTATIONS

H_v-structures used in Representation (abbr. rep) Theory of H_v-groups can be achieved by generalized permutations, using translations [38], or by H_v-matrices [31],[33],[39],[40],[45],[47],[60].

H_v-matrix (or h/v-matrix) is a matrix with entries of an H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type $m\times n$ and $n\times r$ respectively, is defined in the usual manner and it is a set of $m\times r$ H_v-matrices. The sum of products of elements of the H_v-ring is considered

to be the n-ary circle hope on the hyper-sum. The hyperproduct of H_v-matrices is not necessarily WASS.

The problem of the H_v-matrix (or h/v-group) reps is the following:

Definition 3.1 Let $(H,\cdot)$ be H_v-group (or h/v-group). Find an H_v-ring (or h/v-ring) $(R,+,\cdot)$, a set $M_R=\{(a_{ij}) \mid a_{ij}\in R\}$ and a map $T:H\to M_R:h\mapsto T(h)$ such that

$$T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\ \forall h_1,h_2\in H.$$

T is an *H_v-matrix (or h/v-matrix) rep.* If $T(h_1h_2)\subset T(h_1)T(h_2)$, $\forall h_1,h_2\in H$, then T is an *inclusion reps.* If $T(h_1h_2)=T(h_1)T(h_2)=\{T(h) \mid h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then T is a *good rep* and an induced rep T* of the hypergroup algebra is obtained. If T is one to one and the good condition is valid then it is called *faithful rep.*

The rep problem is complicated mainly because the cardinality of the product of two H_v-matrices (or h/v-matrices) is very big. The problem can be simplified in several special cases such as if the h/v-rings have 0 and 1 and, even more, if they are scalars.

The main theorem of the theory of reps is the following [39], [40]:

Theorem 3.2 A necessary condition in order to have an inclusion rep T of an h/v-group $(H,\cdot)$ by n×n, h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following: $\forall\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a))\subset\{A=(a'_{ij}) \mid a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}$$

Thus, inclusion rep $T:H\to M_R:a\mapsto T(a)=(a_{ij})$ induces an homomorphic T* of H/β^* on R/γ^* by $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of $T^*(\beta^*(a))$. T* is called *fundamental induced rep* of T.

Let T be rep of an h/v-group H by h/v-matrices and let $tr_\varphi(T(x))=\gamma^*(Tx_{ii})$ be the fundamental trace, then is called *fundamental character*, the mapping

$$\chi_T:H\to R/\gamma^*:\ x\mapsto\chi_T(x)=tr_\varphi(T(x))=trT^*(x).$$

In reps of H_v-groups there are two difficulties: First to find an H_v-ring or an H_v-field and second, to find an appropriate set of H_v-matrices. However more interesting are the small H_v-fields where the results have one or few elements.

We present the Lie-Santilli admissibility on square matrices [28], [59].

Construction 3.3 Let $(L=M_{m,n},+)$ be a H_v-vector space of m×n hyper-matrices over the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of F/γ^*. Similarly, let ω_L be the core of the canonical map $\varphi':L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Take any two

subsets $R,S \subseteq L$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hope:

$$[\,,\,]_{RS}: L\times L \to P(L): [x,y]_{RS}= xR^{t}y-yS^{t}x.$$

Notice that $[x,y]_{RS}=xR^{t}y-yS^{t}x=\{xr^{t}y-ys^{t}x \mid r\in R \text{ and } s\in S\}$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) $R=\{e\}$ then $[x,y]_{RS} = xy-yS^{t}x = \{xy-ys^{t}x \mid s\in S\}$

(b) $S=\{e\}$ then $[x,y]_{RS} = xR^{t}y-yx = \{xr^{t}y-yx \mid r\in R\}$

(c) $R=\{r_1,r_2\}$ and $S=\{s_1,s_2\}$ then

$$[x,y]_{RS} = xR^{t}y-yS^{t}x = \{xr_1{}^{t}y-ys_1{}^{t}x,\ xr_1{}^{t}y-ys_2{}^{t}x,\ xr_2{}^{t}y-ys_1{}^{t}x,\ xr_2{}^{t}y-ys_2{}^{t}x\}$$

4. THE ISO-HYPERNUMBERS AND APPLICATIONS

Last decades H_v-structures have applications in other branches of mathematics and in other sciences. Applications range from biomathematics - conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [1], [3],[11],[12],[13],[14],[24],[25],[26],[27],[29],[51],[53],[56],[58],[61],[62].

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting' of the n-dimensional trivial unit matrix into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit [9]. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1996 [27] and they are called *e-hyperfields*.

Definition 4.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 4.2 A hyperstructure $(F,+,\cdot)$, where $(+)$ is operation and $(\cdot)$ a hope, is called *e-hyperfield* if the following axioms are valid: $(F,+)$ is abelian group with unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and $\forall x\in F$ there is a unique inverse x^{-1}: $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 4.3 *Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes (Ä) as follows:

$$x\ddot{A}y = \{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\}, \text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$g_1, g_2,\ldots$ are not necessarily the same for each pair (x,y). (G,Ä) is an H_v-group, in fact it is an H_b-group which contains the $(G,\cdot)$. (G,Ä) is an e-hypergroup. Moreover, if ∀x,y such that xy=e, so we have xÄy=xy, then (G,Ä) becomes a strong e-hypergroup.

The main e-construction gives an extremely large number of e-hopes.

Example. Take the quaternions $\boldsymbol{Q}=\{1,-1, i,-i, j,-j, k,-k\}$, with $i^2=j^2=-1$, ij=-ji=k, and denote $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$. We define hopes (∗) by enlarging few products. For example, (-1)∗k=$\underline{k}$, k∗i=$\underline{j}$ and i∗j=$\underline{k}$. Then the hyperstructure (Q,∗) is a strong e-hypergroup.

According to Santilli's iso-theory, on a field F=(F,+,·), *a general isofield* $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where a∈F, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b},\ \text{ with }\ \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}} \qquad \text{(i)}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of Fé,

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\times\hat{1}=\hat{a},\ \forall\hat{a}\in\hat{\boldsymbol{F}} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously [25].

Construction 4.4 *General enlargement.* On a field $\boldsymbol{F}=(\boldsymbol{F},+,\cdot)$ and the isofield $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ we replace $\hat{T}$ in the results of the iso-product

$$\hat{a}\,\hat{\times}\,b = \hat{a}\times\hat{T}\times\hat{b},\ \text{ with }\ \hat{1}=\hat{T}^{-1}$$

tby a set $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{\boldsymbol{F}}$, containing $\hat{T}$, in hyperproducts $\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{\boldsymbol{F}}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore the new iso-hope is

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{H}_{ab}\times\hat{b} = \hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}} \qquad \text{(iii)}$$

$\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ becomes *isoH$_v$-field.* The elements of $\boldsymbol{F}$ are called *isoH$_v$-numbers* or *isonumbers*.

Remark. More important hopes, of the above construction, are the ones where only for few ordered pairs $(\hat{a},\hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few, preferable exactly one. Thus, this special case is if there exists only one pair (âé,bé) for which $\hat{a} \hat{\times} \hat{b} = \hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b}$, $\forall \hat{a},\hat{b}\in\hat{F}$ then we have a hyperstructure called *very thin isoH$_v$-field* [27], [58].

Construction 4.5 *The P-hope*. Consider an isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ with $\hat{a}=a\times\hat{1}$, the isonumbers, where a∈F, and $\hat{1}$ is a positive-defined element generally outside F, with two operations $(\hat{+})$ and $(\hat{\times})$, where $(\hat{+})$ is the sum with the conventional additive unit 0, and $(\hat{\times})$ is the iso-multiplication

$$\hat{a} \hat{\times} \hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \text{ with } \hat{1} = \hat{T}^{-1}, \forall \hat{a},\hat{b}\in\hat{F}$$

Take a set $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, with $\hat{p}_1,\ldots, \hat{p}_s\in\hat{F}-\{\hat{0},\hat{1}\}$, define the ***iso-P-H$_v$-field***, $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times}_P)$ where the hope $\hat{\times}_P$ is defined as follows:

$$\hat{a} \hat{\times}_P \hat{b} := \begin{cases} \hat{a}\times\hat{P}^\wedge\times\hat{b} = \{\hat{a}\times\hat{h}^\wedge\times\hat{b} \mid \hat{h}^\wedge\in\hat{P}^\wedge\} & \text{if } \hat{a}\neq\hat{1} \text{ and } \hat{b}\neq\hat{1} \\ \hat{a}\times\hat{T}^\wedge\times\hat{b} & \text{if } \hat{a}=\hat{1} \text{ or } \hat{b}=\hat{1} \end{cases} \quad \text{(iv)}$$

The elements of $\hat{F}$ are called *iso-P-H$_v$-numbers*.

Remarkable is when $\hat{P}=\{\hat{T},\hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. The inverses in iso-P-H$_v$-fields, are not necessarily unique.

Example. Non degenerate examples on the above on the small field $(\boldsymbol{Z}_7,+,\cdot)$. In the $\hat{\boldsymbol{Z}}_7=\hat{\boldsymbol{Z}}_7(\hat{a},\hat{+},\hat{\times})$, where we denote $\hat{\boldsymbol{Z}}_7=\{\underline{\hat{0}}\underline{\hat{1}},\underline{\hat{2}},\underline{\hat{3}},\underline{\hat{4}},\underline{\hat{5}},\underline{\hat{6}}\}$, the weak associative multiplicative hope is described by only one enlargement $\underline{\hat{2}} \hat{\times} \underline{\hat{6}} = \{\underline{\hat{2}},\underline{\hat{5}}\}$

5. SMALL HYPERNUMBERS. SMALL h/v-FIELDS

We present now the *small non-degenerate h/v-fields* on $(\boldsymbol{Z}_n,+,\cdot)$ which satisfy the following conditions, appropriate in Santilli's iso-theory:

1. *very thin minimal,*
2. *COW (non-commutative),*
3. *they have the elements 0 and 1, scalars,*
4. *if an element has inverse element, this is unique.*

Remark that last condition means than we cannot enlarge the result if it is 1 and we cannot put 1 in enlargement. Moreover we study only the upper

triangular cases since the corresponding under triangular are isomorphic since the commutativity is valid for the underline rings.

Theorem 5.1 All multiplicative h/v-fields defined on $(\boldsymbol{Z}_4,+,\cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:
The only product which is set is $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$.
Fundamental classes: $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have $(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

Example. Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular h/v-matrices on the above h/v-field $(\boldsymbol{Z}_4,+,\otimes)$ of the case that only $2\otimes 3=\{0,2\}$ is a hyperproduct:

$$I=E_{11}+E_{22},\ a=E_{11}+E_{12}+E_{22},\ b=E_{11}+2E_{12}+E_{22},\ c=E_{11}+3E_{12}+E_{22},$$

$$d=E_{11}+3E_{22},\ e=E_{11}+E_{12}+3E_{22},\ f=E_{11}+2E_{12}+3E_{22},\ g=E_{11}+3E_{12}+3E_{22},$$

then, we obtain for the set $\boldsymbol{X}=\{I,a,b,c,d,e,f,g\}$, that the $(\boldsymbol{X},\otimes)$ is non-COW, H_v-group and we can prove that the fundamental classes are the $\underline{a}=\{a,c\}$, $\underline{d}=\{d,f\}$, $\underline{e}=\{e,g\}$ and the fundamental group is isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$. In this H_v-group there is only one unit and every element has a unique double inverse. Only f has one more right inverse element, the d, since $f\otimes d=\{I,b\}$. We obtain that the $(\boldsymbol{X},\otimes)$ is not cyclic.

Theorem 5.2 All multiplicative h/v-fields defined on $(\boldsymbol{Z}_6,+,\cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:
We have the only one hyperproduct,

(I) $2\otimes 3=\{0,3\}$, $2\otimes 4=\{2,5\}$, $3\otimes 4=\{0,3\}$, $3\otimes 5=\{0,3\}$, $4\otimes 5=\{2,5\}$

In all these cases the fundamental classes are $[0]=\{0,3\}$, $[1]=\{1,4\}$, $[2]=\{2,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_3,+,\cdot)$.

(II) $2\otimes 3=\{0,2\}$ or $2\otimes 3=\{0,4\}$, $2\otimes 4=\{0,2\}$ or $\{2,4\}$, $2\otimes 5=\{0,4\}$ or $2\otimes 5=\{2,4\}$, $3\otimes 4=\{0,2\}$ or $\{0,4\}$, $3\otimes 5=\{3,5\}$, $4\otimes 5=\{0,2\}$ or $\{2,4\}$.

In all these cases the fundamental classes are $[0]=\{0,2,4\}$, $[1]=\{1,3,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

Example. In the case of the h/v-field $(\boldsymbol{Z}_6,+,\otimes)$ where the only one hyperproduct is $2\otimes 4=\{2,5\}$ take the h/v-matrices of type $\underline{i}=E_{11}+iE_{12}+4E_{22}$, where $i=0,1,\ldots,5$, then the multiplicative table of the hyperproduct of those h/v-matrices is

$\otimes$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{1}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{2}$	$\underline{2}$	$\underline{0,3}$	$\underline{1,4}$	$\underline{2,5}$	$\underline{0,3}$	$\underline{1,4}$
$\underline{3}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$
$\underline{4}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
$\underline{5}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{0}$	$\underline{1}$

where the fundamental classes are $(0)=\{\underline{0},\underline{3}\}$, $(1)=\{\underline{1},\underline{4}\}$, $(\underline{2})=\{\underline{2},\underline{5}\}$ and the fundamental group is isomorphic to $(\boldsymbol{Z}_3,+)$. Remark that $(\boldsymbol{Z}_6,\otimes)$ is an h/v-group which is cyclic where the elements $\underline{2}$ and $\underline{4}$ are generators of period 4.

Theorem 5.3 All multiplicative h/v-fields defined on $(\boldsymbol{Z}_9,+,\cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:

We have the only one hyperproduct,

$2\otimes3=\{0,6\}$ or $\{3,6\}$, $2\otimes4=\{2,8\}$ or $\{5,8\}$, $2\otimes6=\{0,3\}$ or $\{3,6\}$,

$2\otimes7=\{2,5\}$ or $\{5,8\}$, $2\otimes8=\{1,7\}$ or $\{4,7\}$, $3\otimes4=\{0,3\}$ or $\{3,6\}$,

$3\otimes5=\{0,6\}$ or $\{3,6\}$, $3\otimes6=\{0,3\}$ or $\{0,6\}$, $3\otimes7=\{0,3\}$ or $\{3,6\}$,

$3\otimes8=\{0,6\}$ or $\{3,6\}$, $4\otimes5=\{2,5\}$ or $\{2,8\}$, $4\otimes6=\{0,6\}$ or $\{3,6\}$,

$4\otimes8=\{2,5\}$ or $\{5,8\}$, $5\otimes6=\{0,3\}$ or $\{3,6\}$, $5\otimes7=\{2,8\}$ or $\{5,8\}$,

$5\otimes8=\{1,4\}$ or $\{4,7\}$, $6\otimes7=\{0,6\}$ or $\{3,6\}$, $6\otimes8=\{0,3\}$ or $\{3,6\}$,

$7\otimes8=\{2,5\}$ or $\{2,8\}$

Then in all the above cases the fundamental classes are

$[0]=\{0,3,6\}$, $[1]=\{1,4,7\}$, $[2]=\{2,5,8\}$, and we have $(\boldsymbol{Z}_9,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_3,+,\cdot)$.

Theorem 5.4 All multiplicative h/v-fields defined on $(\boldsymbol{Z}_{10},+,\cdot)$, which have non-degenerate fundamental field, and satisfy the above 4 conditions, are the following isomorphic cases:

(I) We have the only one hyperproduct,

$2\otimes4=\{3,8\}$, $2\otimes5=\{0,5\}$, $2\otimes6=\{2,7\}$, $2\otimes7=\{4,9\}$, $2\otimes9=\{3,8\}$, $3\otimes4=\{2,7\}$,

$3\otimes5=\{0,5\}$, $3\otimes6=\{3,8\}$, $3\otimes8=\{4,9\}$, $3\otimes9=\{2,7\}$, $4\otimes5=\{0,5\}$, $4\otimes6=\{4,9\}$,

$4\otimes7=\{3,8\}$, $4\otimes8=\{2,7\}$, $5\otimes6=\{0,5\}$, $5\otimes7=\{0,5\}$, $5\otimes8=\{0,5\}$, $5\otimes9=\{0,5\}$,

$6\otimes 7=\{2,7\}$, $6\otimes 8=\{3,8\}$, $6\otimes 9=\{4,9\}$, $7\otimes 9=\{3,8\}$, $8\otimes 9=\{2,7\}$.

In all these cases the fundamental classes are

$[0]=\{0,5\}$, $[1]=\{1,6\}$, $[2]=\{2,7\}$, $[3]=\{3,8\}$, $[4]=\{4,9\}$ and we have

$$(Z_{10},+,\otimes)/\gamma^{*}\cong(Z_5,+,\cdot).$$

(II) The cases where we have two classes $[0]=\{0,2,4,6,8\}$ and $[1]=\{1,3,5,7,9\}$, thus the fundamental field is $(Z_{10},+,\otimes)/\gamma^{*}\cong(Z_2,+,\cdot)$, and described as follows:

Taking in the multiplicative table only the results above the diagonal, we enlarge each of the products by putting one element of the same class of the results. We do not enlarge setting the element 1, and we cannot enlarge only the product $3\otimes 7=1$. The number of those h/v-fields is 103.

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HELIX-HOPES ON FINITE H_V-FIELDS[1]

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Abstract

Hyperstructure theory can overcome restrictions which ordinary algebraic structures have. A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. We study the helix-hyperstructures on the representations and we extend our study up to Lie-Santilli theory by using ordinary fields. The related theory can be faced by defining the hyperproduct on the set of non square matrices. The main tools of the Hyperstructure Theory are the fundamental relations which connect the largest class of hyperstructures, the H_V-structures, with the corresponding classical ones. We focus on four dimensional helix-hyperstructures.

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1. INTRODUCTION

We deal with the largest class of hyperstructures called *H_v-structures* introduced in 1990 [11], [13], which satisfy the *weak axioms* where the non-empty intersection replaces the equality.
Basic definitions:

Definitions 1.1 In a set H equipped with a ***hyperoperation*** (which we abbreviate it by ***hope***)

$$\cdot : H\times H\to P(H)-\{\varnothing\}: (x,y)\to x\cdot y\subset H$$

we abbreviate by

WASS the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by

COW the *weak commutativity*: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called H_v-semigroup if it is WASS and is called ***H_v-group*** if it is reproductive H_v-semigroup: $xH=Hx=H$, $\forall x\in H$.
$(R,+,\cdot)$ is called ***H_v-ring*** if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is weak distributive with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\ \forall x,y,z\in R.$$

For more definitions, results and applications on H_v-structures, see books [1],[4],[13] and the survey papers [2],[3],[12],[14],[20],[21]. An extreme class is the following [25]: An H_v-structure is *very thin* iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which $ab=A$, with $\text{card}A>1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

The fundamental relations β^* and γ^* are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [10], [11], [12], [15], [17], [19]. The main theorem is the following:

Theorem 1.2 Let $(H,\cdot)$ be an H_v-group and let us denote by U the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the fundamental relation β^* is the transitive closure of the relation β.

An element is called *single* if its fundamental class is a singleton.

Motivation for H_v-structures:

The quotient of a group with respect to an invariant subgroup is a group. Marty states that, the quotient of a group by any subgroup is a hypergroup. Now, the quotient of a group with respect to any partition is an H_v-group.

Definition 1.3 Let $(H,\cdot)$, $(H,\otimes)$ be H_v-semigroups defined on the same H. $(\cdot)$ is *smaller* than $(\otimes)$, and $(\otimes)$ *greater* than $(\cdot)$, iff there exists automorphism

$$f\in Aut(H,\otimes) \text{ such that } xy\subset f(x\otimes y),\ \forall x\in H.$$

Then $(H,\otimes)$ *contains* $(H,\cdot)$ and write $\cdot\leq\otimes$. If $(H,\cdot)$ is structure, then it is *basic* and $(H,\otimes)$ is an *H_b-structure*.

The Little Theorem 1.4 [13]. Greater hopes of the ones which are WASS or COW, are also WASS and COW, respectively.

Fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ^*:

Definition 1.5 [11], [14]. The H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if the quotient R/γ^* is a field.

Let ω^* be the kernel of the canonical map from R to R/γ^*; then we call *reproductive H_v-field* any H_v-field $(R,+,\cdot)$ if

$$x(R-\omega^*)=(R-\omega^*)x=R-\omega^*,\forall x\in R-\omega^*.$$

From this definition, a new class is introduced [16], [17], [20]:

Definition 1.6 The H_v-semigroup $(H,\cdot)$ is ***h/v-group*** if the H/β^* is a group.

Similarly *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces*, are defined. The h/v-group is a generalization of the H_v-group since the reproductivity is not necessarily valid. Sometimes a kind of *reproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes $\sigma(x)$, then the quotient is reproductive $x\sigma(y)=\sigma(xy)=\sigma(x)y,\ \forall x\in H$.

An H_v-group is called *cyclic* [9], [13], if there is element, called *generator*, which the powers have union the underline set, the minimal power with this property is the *period* of the generator. If there exists an element and a special power, the minimum one, is the underline set, then the H_v-group is called *single-power cyclic*.

Definitions 1.7 [13], [16], [19] Let $(R,+,\cdot)$ be an H_v-ring, $(M,+)$ be COW H_v-group and there exists an external hope $\cdot: R\times M\to P(M):(a,x)\to ax$, such that, $\forall a,b\in R$ and $\forall x,y\in M$ we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\quad (a+b)x\cap(ax+bx)\neq\varnothing,\quad (ab)x\cap a(bx)\neq\varnothing,$$

then M is called an ***H_v-module*** over R. In the case of an H_v-field F instead of H_v-ring R, then the ***H_v-vector space*** is defined.

Definition 1.8 [19] Let (L,+) be H_v-vector space on (F,+,·), φ:F→F/γ^*, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map φ': L→L/ε^* and denote again 0 the zero of L/ε^*. Consider the *bracket* (commutator) hope:

$$[\,,\,]: L\times L\to P(L):\ (x,y)\to[x,y]$$

then L is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$

$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing$, $\forall x,x_1,x_2,y,y_1,y_2\in L$ and $\lambda_1,\lambda_2\in F$

(L2) $[x,x]\cap\omega_L\neq\varnothing$, $\forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing$, $\forall x,y\in L$

A well known and large class of hopes is given as follows [1], [4], [13]:

Definitions 1.9 Let (G,·) be a groupoid, then for every subset P⊂G, P≠∅, we define the following hopes, called ***P-hopes***: $\forall x,y\in G$

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),$$

$$\underline{P}_r: x\underline{P}_ry=(xy)P\cup x(yP),\qquad \underline{P}_l: x\underline{P}_ly=(Px)y\cup P(xy).$$

The (G,$\underline{P}$), (G,$\underline{P}_r$) and (G,$\underline{P}_l$) are called ***P-hyperstructures***. The usual case is for semigroup (G,·), then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and (G,$\underline{P}$) is a semihypergroup but we do not know about (G,$\underline{P}_r$) and (G,$\underline{P}_l$). In some cases, depending on the choice of P, the (G,$\underline{P}_r$) and (G,$\underline{P}_l$) can be associative or WASS.

A generalization of P-hopes is the following [20]: Let (G,·) be an abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:

$$x\times_P y=\begin{cases} x\cdot P\cdot y=\{x\cdot h\cdot y\mid h\in P\} & \text{if } x\neq e \text{ and } y\neq e\\ x\cdot y & \text{if } x=e \text{ or } y=e\end{cases}$$

we call this hope, *P_e-hope*. The hyperstructure (G,$\times_P$) is an abelian H_v-group.

2. THE ISO-HYPERNUMBERS

Last decades H_v-structures have applications in other branches of mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [6], [7], [18], [19], [22].

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit [6]. The *isofields* needed in this theory correspond into the hyperstructures introduced by Santilli & Vougiouklis in 1999 [7] and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology.

Definition 2.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 2.2 A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and $\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 2.3 *The Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\ddot{A})$ as follows:

$$x\ddot{A}y = \{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\},\text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$g_1, g_2,\ldots$ are not necessarily the same for each pair (x,y). $(G,\ddot{A})$ is an H_v-group, it is an H_b-group which contains the $(G,\cdot)$. $(G,\ddot{A})$ is an e-hypergroup. Moreover, if for each x,y such that $xy=e$, so we have $x\ddot{A}y=xy$, then $(G,\ddot{A})$ becomes a strong e-hypergroup.

The main e-construction gives an extremely large number of e-hopes.

Example 2.4 Consider the quaternion group $Q=\{1,-1, i,-i, j,-j, k,-k\}$ with defining relations $i^2 = j^2 = -1$, $ij = -ji = k$. Denoting $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$ we may define a very large number $(*)$ hopes by enlarging only few products. For example, $(-1)*k=\underline{k}$, $k*i=\underline{j}$ and $i*j=\underline{k}$. Then the hyperstructure $(Q,*)$ is a strong e-hypergroup.

According to Santilli's iso-theory, on a field $F=(F,+,\cdot)$, *a general isofield* $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1}, \ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(i)}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of Fé,

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\times\hat{1}=\hat{a}, \ \forall\hat{a}\in\hat{F} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously [7].

In order to transfer this theory into the hyperstructure case we generalize only the new multiplication $\hat{\times}$ from (i), by replacing with a hope including the old one. We introduce two general constructions on this direction as follows:

Construction 2.5 *General enlargement.* On a field $F=(F,+,\cdot)$ and on the isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ we replace in the results of the iso-product

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}$, containing $\hat{T}$, $\forall\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore the new iso-hope is

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{H}_{ab}\times\hat{b} = \hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b}, \ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(iii)}$$

$\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ becomes *isoH$_v$-field.* The elements of F are called *isoH$_v$-numbers* or *isonumbers*.

Remarks 2.6

(i) More important hopes are the ones where only for few ordered pairs $(\hat{a},\hat{b})$ the result is enlarged, even more, the elements $\hat{x}_i$, are only few, preferable

exactly one. Thus, this special case is if there exists only one pair (âé,bé) for which

$$\hat{a}\,\hat{\times}\,\hat{b}=\ \hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\quad \forall\hat{a},\hat{b}\in\hat{F}$$

and the rest are ordinary results, then we have a *very thin isoH$_v$-field* [6], [22].
(ii) The assumption that $\hat{H}_{ab}=\{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, with that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give the isoH$_v$-field with a scalar absorbing $\hat{0}$, a scalar $\hat{1}$, and $\forall\hat{a}\in\hat{F}$, has one inverse.

3. H$_v$-MATRIX REPRESENTATIONS

H$_v$-structures are used in Representation Theory of H$_v$-groups which can be achieved by generalized permutations or by H$_v$-matrices [9], [13], [16], [21].

H$_v$-matrix (or h/v-matrix) is a matrix with entries of an H$_v$-ring or H$_v$-field (or h/v-ring or h/v-field). The hyperproduct of two H$_v$-matrices (a_{ij}) and (b_{ij}), of type m×n and n×r respectively, is defined in the usual manner and it is a set of m×r H$_v$-matrices. The sum of products of elements of the H$_v$-ring is considered to be the n-ary circle hope on the hyperaddition. The hyperproduct of H$_v$-matrices is not necessarily WASS.
The problem of the H$_v$-matrix (or h/v-group) representations is the following:

Definition 3.1 Let $(H,\cdot)$ be H$_v$-group (or h/v-group). Find an H$_v$-ring (or h/v-ring) $(R,+,\cdot)$, a set $M_R=\{(a_{ij})\,|\,a_{ij}\in R\}$ and a map $T:H\to M_R$: $h\mapsto T(h)$ such that

$$T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\ \forall h_1,h_2\in H.$$

T is *H$_v$-matrix (or h/v-matrix) representation.* If $T(h_1h_2)\subset T(h_1)T(h_2)$, $\forall h_1,h_2\in H$, then T is called *inclusion.* If $T(h_1h_2)=T(h_1)T(h_2)=\{T(h)\,|\,h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then T is *good* and then an induced representation T* for the hypergroup algebra is obtained. If T is one to one and good then it is *faithful.*

The main theorem on representations is [13]:

Theorem 3.2 A necessary condition to have an inclusion representation T of an h/v-group $(H,\cdot)$ by n×n, h/v-matrices over the h/v-ring $(R,+,\cdot)$ is the following: For all classes $\beta^*(x)$, $x\in H$ must exist elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a))\subset\{A=(a'_{ij})\,|\,a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}$$

Inclusion $T:H\to M_R:a\mapsto T(a)=(a_{ij})$ induces homomorphic representation T^* of H/β^* on R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. T^* is called *fundamental induced* of T.

In representations, several new classes are used:

Definition 3.3 Let $M=M_{m,n}$ be the module of m×n matrices over R and $P=\{P_i:i\in I\}\subseteq M$. We define a P-hope $\underline{P}$ on M as follows

$$\underline{P}: M\times M\to P(M): (A,B)\ A\underline{P}B=\{AP^t_iB: i\in I\}\subseteq M$$

where P^t denotes the transpose of P.

The hope $\underline{P}$ is bilinear map, is strong associative and inclusion distributive:

$$A\underline{P}(B+C)\subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C\in M$$

Definition 3.4 Let $M=M_{m,n}$ the m×n matrices over R and let us take sets

$$S=\{s_k:k\in K\}\subseteq R,\ Q=\{Q_i:j\in J\}\subseteq M,\ P=\{P_i:i\in I\}\subseteq M.$$

Define three hopes as follows

$$\underline{S}: R\times M\to P(M): (r,A)\to r\underline{S}A=\{(rs_k)A: k\in K\}\subseteq M$$

$$\underline{Q}_+: M\times M\to P(M): (A,B)\to A\underline{Q}_+B=\{A+Q_j+B: j\in J\}\subseteq M$$

$$\underline{P}: M\times M\to P(M): (A,B)\to A\underline{P}B=\{AP^t_iB: i\in I\}\subseteq M$$

Then $(M,\underline{S},\underline{Q}_+,\underline{P})$ is hyperalgebra on R called *general matrix P-hyperalgebra.*

4. SMALL HYPERNUMBERS

In representations important role are playing the small hypernumbers [20].

Construction 4.1 On the ring $(\boldsymbol{Z}_4,+,\cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,

(a) very thin minimal,
(b) COW (non-commutative),
(c) they have the elements 0 and 1, scalars.

Then, we have only the following isomorphic cases $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$. Fundamental classes: $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have $(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

Thus it is isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$. In this H_v-group there is only one unit and every element has a unique double inverse. Only f has one more right inverse element, the d, since $f\otimes d=\{I,b\}$. Moreover, the $(\boldsymbol{X},\otimes)$ is not cyclic.

Construction 4.2 On the ring $(\boldsymbol{Z}_6,+,\cdot)$ we define, up to isomorphism, all multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are:

(a) very thin minimal, i.e. only one product has exactly two elements
(b) COW (non-commutative)
(c) they have the elements 0 and 1, scalars

Then we have the following cases, by giving the only one hyperproduct,

(I) $2\otimes3=\{0,3\}$ or $2\otimes4=\{2,5\}$ or $2\otimes5=\{1,4\}$
$3\otimes4=\{0,3\}$ or $3\otimes5=\{0,3\}$ or $4\otimes5=\{2,5\}$
In all 6 cases the fundamental classes are $[0]=\{0,3\}$, $[1]=\{1,4\}$, $[2]=\{2,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_3,+,\cdot)$.

(II) $2\otimes3=\{0,2\}$ or $2\otimes3=\{0,4\}$ or $2\otimes4=\{0,2\}$ or $2\otimes4=\{2,4\}$ or
$2\otimes5=\{0,4\}$ or $2\otimes5=\{2,4\}$ or $3\otimes4=\{0,2\}$ or $3\otimes4=\{0,4\}$ or
$3\otimes5=\{1,3\}$ or $3\otimes5=\{3,5\}$ or $4\otimes5=\{0,2\}$ or $4\otimes5=\{2,4\}$.
In all 12 cases the fundamental classes are $[0]=\{0,2,4\}$, $[1]=\{1,3,5\}$ and we have $(\boldsymbol{Z}_6,+,\otimes)/\gamma^* \cong (\boldsymbol{Z}_2,+,\cdot)$.

Remark that if we need h/v-fields where the elements have at most one inverse element, then we must exclude the case of $2\otimes5=\{1,4\}$ from (I), and the case $3\otimes5=\{1,3\}$ from (II).

5. HELIX-HOPES AND APPLICATIONS

Recall some definitions from [5], [8], [23], [24]:

Definition 5.1 Let $A=(a_{ij})\in M_{m,n}$ be m×n matrix and $s,t\in N$ be naturals such that $1\le s\le m$, $1\le t\le n$. We define the map $\underline{cst}$ from $M_{m,n}$ to $M_{s,t}$ by corresponding to the matrix A, the matrix $A\underline{cst}=(a_{ij})$ where $1\le i\le s$, $1\le j\le t$. We call this map *cut-projection* of type $\underline{st}$. Thus $A\underline{cst}$ is matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.

We use cut-projections on all types of matrices to define sums and products.

Definitions 5.2 Let $A=(a_{ij})\in M_{m,n}$ be an m×n matrix and $s,t\in N$, such that $1\le s\le m$, $1\le t\le n$. We define the mod-like map $\underline{st}$ from $M_{m,n}$ to $M_{s,t}$ by corresponding to A the matrix $A\underline{st}=(\underline{a}_{ij})$ which has as entries the sets

$$\underline{a}_{ij} = \{a_{i+\kappa s,j+\lambda t} \mid 1\le i\le s,\ 1\le j\le t. \text{ and } \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n\}.$$

Thus we have the map

$$\underline{st}: M_{m,n} \to M_{s,t}: A \to A\underline{st} = (\underline{a}_{ij}).$$

We call this multivalued map *helix-projection* of type $\underline{st}$. $A\underline{st}$ is a set of s×t-matrices $X=(x_{ij})$ such that $x_{ij}\in\underline{a}_{ij}$, $\forall i,j$. Obviously $A\underline{mn}=A$.
Let $A=(a_{ij})\in M_{m\times n}$ be a matrix and $s,t\in N$ such that $1\le s\le m$, $1\le t\le n$. Then it is clear that we can apply the helix-projection first on the rows and then on the columns, the result is the same if we apply the helix-progection on both, rows and columns. Therefore we have

$$(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}.$$

Let $A=(a_{ij})\in M_{m\times n}$ be matrix and $s,t\in N$ such that $1\le s\le m$, $1\le t\le n$. Then if $A\underline{st}$ is not a set but one single matrix then we call A *cut-helix matrix* of type s×t. In other words the matrix A is a helix matrix of type s×t, if $A\underline{cst}= A\underline{st}$.

Definitions 5.3
(a) Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$ be matrices and $s=\min(m,u)$, $t=\min(n,u)$. We define a hope, called *helix-addition* or ***helix-sum***, as follows:

$$\oplus: M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}): (A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where

$$(\underline{a}_{ij})+(\underline{b}_{ij})= \{(c_{ij})= (a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

(b) Let $A=(a_{ij})\in M_{m\times n}$ and $B=(b_{ij})\in M_{u\times v}$ be matrices and $s=\min(n,u)$. We define a hope, called *helix-multiplication* or ***helix-product***, as follows:

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}):(A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where

$$(\underline{a}_{ij})\cdot(\underline{b}_{ij})= \{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij}\}.$$

The helix-sum is external hope since it is defined on different sets and the result is also in different set. The commutativity is valid in the helix-sum. For the helix-product we remark that we have $A\otimes B=A\underline{ms}\cdot B\underline{sv}$ so we have either $A\underline{ms}=A$ or $B\underline{sv}=B$, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. In $M_{m\times n}$ the addition is ordinary operation, thus we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the ***helix-Lie Algebra*** is defined [19], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ε^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $M_{m,n}$ where $m<n$. We have analogous results if $m>n$ and for $m=n$ we have the classical theory.

Notation. For given $\kappa\in\mathbb{N}-\{0\}$, we denote by $\underline{\kappa}$ the remainder resulting from its division by m if the remainder is non zero, and $\underline{\kappa}=m$ if the remainder is zero. Thus a matrix $A=(a_{\kappa\lambda})\in M_{m,n}$, $m<n$ is a *cut-helix matrix* if we have $a_{\kappa\lambda}=a_{\kappa\underline{\lambda}}$, $\forall\kappa,\lambda\in\mathbb{N}-\{0\}$.

Moreover let us denote by $I_c=(c_{\kappa\lambda})$ the *cut-helix unit matrix* which the cut matrix is the unit matrix I_m. Therefore, since $I_m=(\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall\kappa,\lambda$, we have $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$.

Proposition 5.4 For $m<n$ in $(M_{m,n},\otimes)$ the cut-helix unit matrix $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.

Proof. Let $A,B\in M_{m,n}$ then in the helix-multiplication, since $m<n$, we take helix projection of the matrix A, therefore, the result $A\otimes B$ is singleton if the matrix A is a cut-helix matrix of type $m\times m$. Moreover, in order to have $A\otimes B=A\underline{mm}\cdot B=B$, the matrix $A\underline{mm}$ must be the unit matrix. Consequently, $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, $\forall\kappa,\lambda\in\mathbb{N}-\{0\}$, is necessarily the left scalar unit.

Let $A=(a_{uv})\in M_{m,n}$ and consider the hyperproduct $A\otimes I_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1\leq\kappa\leq m$, $1\leq\lambda\leq n$, of the form

$$\sum\underline{a}_{\kappa s}c_{s\lambda}=\sum\underline{a}_{\kappa s}\delta_{s\underline{\lambda}}=\underline{a}_{\kappa\underline{\lambda}}\ni a_{\kappa\lambda}.$$

Therefore $A\otimes I_c\ni A$, $\forall A\in M_{m,n}$. ■

In the following examples of the helix-hope, we denote E_{ij} any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Example 5.5 Consider the 2×3 matrices of the forms,

$$A_{\kappa\lambda}=E_{11}+E_{13}+\kappa E_{21}+E_{22}+\lambda E_{23},\ \forall\kappa,\lambda\in\mathbb{Z}.$$

Then we obtain $A_{\kappa\lambda}\otimes A_{st}=\{A_{\kappa+s,\kappa+t},A_{\kappa+s,\lambda+t},A_{\lambda+s,\kappa+t},A_{\lambda+s,\lambda+t}\}$.

Moreover $A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+s},A_{\kappa+s,\lambda+t},A_{\kappa+t,\lambda+s},A_{\kappa+t,\lambda+t}\}$, so

$$A_{\kappa\lambda}\otimes A_{st}\cap A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+t}\},\text{ thus }(\otimes)\text{ is COW.}$$

The helix multiplication $(\otimes)$ is associative.

Example 5.6 Consider all *traceless* matrices $A=(a_{ij})\in M_{2,3}$, in the sence that $a_{11}+a_{22}=0$. In this case, the cardinality of the helix-product of any two matrices is 1, or 2^3, or 2^6. These correspond to the cases: $a_{11}=a_{13}$ and $a_{21}=a_{23}$, or only

$a_{11}=a_{13}$ either only $a_{21}=a_{23}$, or if there is no restriction, respectively. For the Lie-bracket of two traceless matrices the corresponding cardinalities are up to 1, or 2^6, or 2^{12}, respectively.

From this example it is obvious the following:

Theorem 5.7 The Lie-bracket of two traceless matrices $A=(a_{ij})$, $B=(b_{ij})\in M_{m\times n}$, $m<n$, contain at least one traceless matrix.

Example 5.8 Let us denote by E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries. Then take the following 2×2 upper triangular h/v-matrices on the above h/v-field $(\boldsymbol{Z}_4,+,\otimes)$, on the set $\boldsymbol{Z}_4=\{0,1,2,3\}$, of the case that only $2\otimes 3=\{0,2\}$ is a hyperproduct:

$$I=E_{11}+E_{22},\quad a=E_{11}+E_{12}+E_{22},\quad b=E_{11}+2E_{12}+E_{22},\quad c=E_{11}+3E_{12}+E_{22},$$

$$d=E_{11}+3E_{22},\quad e=E_{11}+E_{12}+3E_{22},\quad f=E_{11}+2E_{12}+3E_{22},\quad g=E_{11}+3E_{12}+3E_{22},$$

A hyper-matrix representation of four dimensional case with helix-hope:

Example 5.9 On the field of real or complex numbers we consider all 3×5 matrices of type

$$\boldsymbol{A}=\begin{pmatrix} 1 & a & b & 1 & d \\ 0 & 1 & c & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

This set is closed under the helix-hope. That means that the helix-product of two such matrices is a 3×5 matrix, of the same type. In fact we have

$$\mathbf{A}\otimes\mathbf{A}'=\begin{pmatrix} 1 & a & b & 1 & d \\ 0 & 1 & c & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}\otimes\begin{pmatrix} 1 & a' & b' & 1 & d' \\ 0 & 1 & c' & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & \{a,d\} & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\bullet\begin{pmatrix} 1 & a' & b' & 1 & d' \\ 0 & 1 & c' & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 1 & \{a+a',\, d+a'\} & \{b+b'+ac',\, b+b'+dc'\} & 1 & \{a+d',\, d+d'\} \\ 0 & 1 & c+c' & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Therefore the result is asset with 8 matrices.

The WASS axiom is valid because we have the following:

Lets denote E_{ij} the matrix with 1 in the ij-entry and zero in the rest entries so

$$\mathbf{A}= E_{11}+aE_{12}+bE_{13}+E_{14}+dE_{15}+E_{22}+cE_{23}+E_{25}+E_{33}$$

Taking three matrices **A**, **A′**, **A″** in hyperproducts **(A⊗A′)⊗A″** and **A⊗(A′⊗A″)**, we, after calculations, obtain that in the entries 12, 15 and 23, respectively, we have the same sets:

{a+a′+a″, a+c′+a″, c+a′+a″, c+c′+a″}, {a+a′+c″, a+c′+c″, c+a′+c″, c+c′+c″}

and {d+d′+d″}

but in the entry 13, we have for the left, the set

{b+b′+b″+ad′+ad″+a′d″, b+b′+b″+ad″+a′d″+cd″, b+b′+b″+ad′+ad″+c′d″,

b+b′+b″+ad″+cd′+c′d″, b+b′+b″+ad′+a′d″+cd″, b+b′+b″+cd′+cd″+a′d″,

b+b′+b″+ad′+cd″+c′d″, b+b′+b″+cd′+cd″+c′d″},

and, on the right, we have the set

{ b+b′+b″+ad′+ad″+a′d″, b+b′+b″+ad′+ad″+c′d″, b+b′+b″+cd′+cd″+a′d″,

b+b′+b″+cd′+cd″+c′d″}.

Therefore, we have **(A⊗A′)⊗A″ ⊃ A⊗(A′⊗A″)**

Examples 5.10

(a) On the same type of matrices using the Construction 4.1, on $(\boldsymbol{Z}_4,+,\cdot)$ we take the small h/v-field $(\boldsymbol{Z}_4,+,\otimes)$, where only $2\otimes 3=\{0,2\}$, where we remind that the fundamental classes are {0,2}, {1,3}. We take from the set of all matrices

$$\boldsymbol{A}=\begin{pmatrix} 1 & a & b & 1 & d \\ 0 & 1 & c & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

the matrix

$$\boldsymbol{X}=\begin{pmatrix} 1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Then the powers of this matrix are

$$X^2 = \begin{pmatrix} 1 & \{0,2\} & \{1,3\} & 1 & \{0,2\} \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$X^3 = \begin{pmatrix} 1 & \{1,3\} & \{1,3\} & 1 & \{1,3\} \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and so on}$$

We obtain that the generating set is the following

$$\begin{pmatrix} 1 & \{0,2\} & Z_4 & 1 & \{0,2\} \\ 0 & 1 & \{0,2\} & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \cup \begin{pmatrix} 1 & \{1,3\} & Z_4 & 1 & \{1,3\} \\ 0 & 1 & \{1,3\} & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Cyclic set but not single-power cyclic.

(b) If we take the matrix

$$Y = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Then the powers of this matrix are

$$Y^2 = \begin{pmatrix} 1 & \{2,3\} & \{1,2\} & 1 & \{0,3\} \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$Y^3 = \begin{pmatrix} 1 & Z_4 & Z_4 & 1 & Z_4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and so on}$$

We obtain that the generating set is the set of all matrices of this type:

$$\begin{pmatrix} 1 & Z_4 & Z_4 & 1 & Z_4 \\ 0 & 1 & Z_4 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

This set is a single-power cyclic of period 6.

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ON THE PROPERTIES OF REGULAR GRAPHS

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Abstract

Let $(P, \leq)$ be a poset with the least element 0. The undirected regular graph of ideals of P, denoted by $\Gamma(P)$, is a graph with all non-trivial ideals of P being the vertex-set and, for two distinct vertices I and J, I is adjacent to J if and only if I contains a non zero-divisor on J or J contains a non zero-divisor on I. In this paper, we study some properties of $\Gamma(P)$.

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1 Introduction

The investigation of graphs associated to semigroups is a large research area. There are many papers which interlink graph theory and poset theory. For example, Cayley graphs have been studied in [11] and [12], zero-divisor graphs have been studied in [5], [6], [7], [10], and cozero-divisor graphs have been introduced and investigated in [2], [3] and [4].

Recently, in [14], Nikmehr and Shaveisi defined the regular digraph of ideals of R, denoted by $\overrightarrow{\Gamma_{\text{reg}}}(R)$, which is a digraph whose vertex-set is the set of all non-trivial ideals of R and, for every two distinct vertices I and J, there is an arc from I to J, denoted by $I \to J$, whenever $I \nsubseteq Z(J)$. In other words, I contains a J-regular element. They studied and investigated some properties of this graph.

Let $(P, \leq)$ be a partially ordered set (poset, briefly) with the least element 0 and I be a non-empty subset of P. We say that I is an ideal of P if, for arbitrary elements x and y in P, the relations $x \in I$ and $y \leqslant x$ imply that $y \in I$. Also, we put

$$Z(I) = \{x \in P \mid \text{there exists a nonzero element } y \in I \text{ such that } \{x, y\}^l = \{0\}\},$$

where $\{x, y\}^l$ is the set of all elements t such that $t \leq x$ and $t \leq y$. Let $I(P)$ be the set of all ideals of P. An element $r \in L$ is called *I-regular* if $r \notin Z(I)$, where $I \in I(P)$.

The regular digraph of ideals of P, denoted by $\overrightarrow{\Gamma}(P)$, which is defined and studied in [1], is a digraph whose vertex-set is the set of all nontrivial ideals of P and, for every two distinct vertices I and J, there is an arc from I to J whenever I contains a J-regular element, or equivallently $I \nsubseteq Z(J)$. Also, the undirected regular graph of ideals of P, denoted by $\Gamma(P)$, is a graph with all nontrivial ideals of P being the vertex-set, and, for two distinct vertices I and J, I is adjacent to J if and only if $I \nsubseteq Z(J)$ or $J \nsubseteq Z(I)$.

In [1], the authors studied some basic properties of $\overrightarrow{\Gamma}(P)$, and also they investigated the planarity of $\Gamma(P)$. In this paper we study some properties of the regular graph of ideals associated to a poset.

Now, we recall some definitions and notations on graphs and partially ordered sets. We use the standard terminology of graphs in [8] and partially ordered sets in [9].

A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets, so that no edges has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same part. A graph G is said to be *planar* if it can be drawn in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, where K_n is the complete graph with n vertices and $K_{m,n}$ is the complete bipartite graph, for positive integers m and n. An undirected graph is an *outerplanar* graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization of outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of the complete graph K_4 or the complete bipartite graph $K_{2,3}$. Clearly, every outerplanar graph is planar.

In a partially ordered set $(P, \leqslant)$ with a least element 0, an element a is called an *atom* if $a \neq 0$, and, for an element x in P, the relation $0 \leqslant x \leqslant a$ implies either $x = 0$ or $x = a$. Assume that S is a subset of P. Then an element x in P is a *lower bound* of S if $x \leqslant s$ for all $s \in S$. An *upper bound* is defined in a dual manner. The set of all lower bounds of S is denoted by S^ℓ and the set of all upper bounds of S by S^u, that is,

$$S^\ell := \{x \in P \mid x \leqslant s, \text{ for all } s \in S\}$$

and

$$S^u := \{x \in P \mid s \leqslant x, \text{ for all } s \in S\}.$$

Now, suppose that I is a non-empty subset of P. A poset P is *atomic*, if every non-zero element is an atom. In this paper, we assume that P is a finite poset and $A(P) = \{a_1, a_2, \ldots, a_n\}$ is the set of all atoms of P.

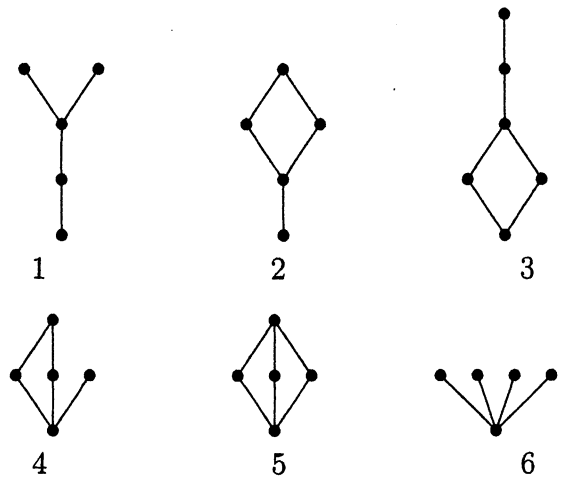
2 Outerplanar regular graph of posets

In this section, we study the outerplanarity of $\Gamma(P)$. Since every outerplanar graph is planar, in order to investigating outerplanar graphs, we should consider planar graphs.

In the following theorem which is from [1] all planar regular graphs of a finite poset is characterized.

Theorem 2.1. *$\Gamma(P)$ is planar if and only if $|P| \leq 4$, or P is a chain with $|P| \in \{5, 6\}$, or P is one of the posets in the following figure.*

Proof. See [1, Theorem 4.4]. □



In the following theorem, we characterize all finite posets with outerplanar regular graphs.

Theorem 2.2. *$\Gamma(P)$ is outerplanar if and only if $|P| \leq 4$, or P is a chain with $|P| = 5$, or P is one of the posets $3, 4$ or 5 in the above figure.*

Proof. First assume that $\Gamma(P)$ is outerplanar. Then clearly it is planar, and so by Theorem 2.1, $|P| \leq 4$, or P is a chain with $|P| \in \{5, 6\}$, or P is one of the posets in the above figure. If P is a chain with six elements, or P is one of the posets 1 or 2 in the above figure, then $\Gamma(P)$ contains the graph K_4, and so it is not outerplanar. If P is the poset 6 in the above figure, then $\Gamma(P)$ contains the graph $K_{2,3}$ with vertex set $\{\{0, a\}, \{0, b\}\} \cup \{\{0, a, b\}, \{0, a, b, c\}\{0, a, b, d\}\}$, where a, b, c, d are the atoms of P, which implies that $\Gamma(P)$ is not outerplanar. Now, if $|P| \leq 4$, or P is a chain with $|P| = 5$, or P is one of the posets $3, 4$ or 5 in the above figure, then one can easily check that $\Gamma(P)$ is outerplanar.

The converse statement is clear. □

3 More properties of $\Gamma(P)$

We begin this section with the following lemma.

Lemma 3.1. *Let P be a poset. Then if $\Gamma(P)$ is a complete bipartite graph, then $|A(P)| \leq 2$.*

Proof. Assume that X and Y are two parts of $\Gamma(P)$. Suppose on the contrary that $|A(P)| \geq 3$. Then $\{0, a_1\}$, $\{0, a_2\}$ and $\{0, a_3\}$ are in the same part, say part X, where $a_1, a_2, a_3 \in A(P)$. Now, if $\{0, a_1, a_2\}$ is in part X, then it is adjacent to $\{0, a_1\}$ and $\{0, a_2\}$, which is impossible. So we have that the vertex $\{0, a_1, a_2\}$ is in part Y. But then $\{0, a_3\}$ is not adjacent to $\{0, a_1, a_2\}$ which is a contradiction. □

Clearly, if $|A(P)| = 1$, then $\Gamma(P)$ is a complete graph. Thus we have the following corollary:

Corollary 3.2. *Let P be a poset and $|A(P)| = 1$. Then the following conditions are equivalent.*

(i) $\Gamma(P)$ is a complete bipartite graph.

(ii) P is a chain with $|P| \leq 4$.

(iii) $\Gamma(P)$ is a star graph.

Proposition 3.3. *Let P be a poset. Then $\Gamma(P)$ is complete bipartite if and only if P is a chain with $|P| \leq 4$, or P is one of the posets in Figure 1.*

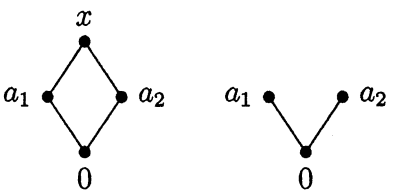


Figure 1

Proof. First suppose that $\Gamma(P)$ is a complete bipartite graph and P is non of the posets in Figure 1. By Lemma 3.1, we have $|A(P)| \leq 2$. By Corollary 3.2, we have $|A(P)| = 2$. Set $A(P) = \{a_1, a_2\}$ and suppose that X and Y are two parts of $\Gamma(P)$. Now, we have the following two cases:

Case 1. There exists an element $x \in \{a_i\}^u \setminus \{a_j\}^u$, where $i \neq j$ and $i, j \in \{1, 2\}$. Without loss of generality, we may assume that $x \in \{a_1\}^u \setminus \{a_2\}^u$. Clearly $\{0, a_1\}$ and $\{0, a_2\}$ belongs to a same part, say X. Then $\{0, a_1, x\}$ and $\{0, a_1, a_2\}$ belongs to Y. But this is impossible, since $\{0, a_1, x\}$ is adjacent to $\{0, a_1, a_2\}$.

Case 2. Suppose that for all $x \in P \setminus \{0, a_1, a_2\}$, we have $x \in \{a_1, a_2\}^u$. In this case the vertices $\{0, a_1\}$ and $\{0, a_2\}$ belong to a same part, say X. Also $\{0, a_1, a_2, x\}$ and $\{0, a_1, a_2\}$ belong to Y, which is again impossible.

Therefore, if $\Gamma(P)$ is a complete bipartite graph, then P is a chain with $|P| \leq 4$, or it is one of the posets in Figure 1.

The converse statement is clear. □

A graph $\Gamma(P)$ is said to be a *split* graph if the vertex-set of $\Gamma(P)$ is a disjoint union of two sets K and S, where K induces a complete subgraph and S is an independent set.

Lemma 3.4. *Let P be a poset such that $\Gamma(P)$ is split. Then $|A(P)| \leq 2$.*

Proof. Let K and S be two parts of $\Gamma(P)$ such that K is complete and S is independent. Assume on the contrary that $|A(P)| \geq 3$. Suppose that $a_1, a_2, a_3 \in A(P)$. If $\{0, a_1\} \in K$, then $\{0, a_2\}$ must be in S, because $\{0, a_1\}$ is not adjacent to $\{0, a_2\}$. Now, we have the following two cases:

Case 1. If $\{0, a_3\}$ is in K, then K is not complete, because $\{0, a_1\}$ is not adjacent to $\{0, a_3\}$.

Case 2. If $\{0, a_3\}$ is in S, then $\{0, a_2, a_3\}$ must be in K, which is a contradiction, because $\{0, a_2, a_3\}$ is not adjacent to $\{0, a_1\}$.

So if $\Gamma(P)$ is split, then $|A(P)| \leq 2$. □

Proposition 3.5. *Let P be a poset. Then $\Gamma(P)$ is split if and only if one of the following conditions holds.*

(i) $|A(P)| = 1$.

(ii) $|A(P)| = 2$ *and* $|P| \leq 4$.

(iii) $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, *for all* $1 \leq i \neq j \leq 2$.

Proof. First suppose that $\Gamma(P)$ is split and $|A(P)| \neq 1$. Then, by Lemma 3.4, we have $|A(P)| = 2$. Now, suppose on the contrary that $|P| \geq 5$ and $\{a_i\}^u \neq \emptyset$, for some $1 \leq i \leq 2$. Then we have the following situations:

There exist distinct elements $x_1, x_2 \in P \backslash \{0, a_1, a_2\}$ such that $x_1, x_2 \in \{a_i\}^u \backslash \{a_j\}^u$, or $x_1 \in \{a_i\}^u \backslash \{a_j\}^u$ and $x_2 \in \{a_1, a_2\}^u$, or $x_1 \in \{a_i\}^u \backslash \{a_j\}^u$ and $x_2 \in \{a_j\}^u \backslash \{a_i\}^u$, for some $1 \leq i \neq j \leq 2$ and $a_1, a_2 \in A(P)$.

So we have that

$$\{\{0, a_1\}, \{0, a_1, x_1\}, \{0, a_1, x_1, x_2, \{0, a_1, a_2\}\} \subseteq K \text{ and } \{\{0, a_2\}\} \subseteq S.$$

Now consider the vertex $u := \{0, a_1, a_2, x_1\}$. If $u \in K$, then u is not adjacent to $\{0, a_1, a_2\}$. So u is in S. But it is impossible, because u is adjacent to $\{0, a_2\}$, and so $\Gamma(P)$ is not split. There fore, $|A(P)| = 1$, or $|A(P)| = 2$ and $|P| \leq 4$ or $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$.

Conversely, if $|A(P)| = 1$, then it is easy to see that $\Gamma(P)$ is split.

Now, if $|A(P)| = 2$, $|P| \leq 4$ or $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$, then we have the following two cases:

Case 1. If there exists an element $x \in \{a_i\}^u \backslash \{a_j\}^u$, for some $1 \leq i \neq j \leq 2$, then

$$K = \{\{0, a_1\}, \{0, a_1, a_2\}, \{0, a_1, x\}\} \text{ and } S = \{\{0, a_2\}\},$$

and so $\Gamma(P)$ is split.

Case 2. If $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$, then

$$K = V(\Gamma(P)) \setminus \{0, a_2\} \text{ and } S = \{\{0, a_2\}\},$$

which means that $\Gamma(P)$ is split.

Now, suppose that $|P| \leq 4$ and $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$. If $|P| = 3$, then clearly $\Gamma(P)$ is split. If $|P| = 4$, then

$$K = V(\Gamma(P)) \setminus \{0, a_2\} \text{ and } S = \{\{0, a_2\}\},$$

and so $\Gamma(P)$ is split. □

In a graph G, a vertex v is called *simplicial* if the subgraph of G induced by the vertex set $\{v\} \cup \mathrm{N}(v)$ is a complete graph.

Proposition 3.6. *Let P be a poset. Then if P is a chain, then all vertices in $\Gamma(P)$ is a simplicial vertex.*

Proof. Suppose that P is a chain. Then the graph $\Gamma(P)$ is a complete graph, and so all vertices are simplicial. □

Proposition 3.7. *Let P be a poset. Then if I is a simplicial vertex, then I has only one atom.*

Proof. Suppose on the contrary that I has at least two atoms. Set $A(P) = \{a_1, a_2\}$. In this case, I is adjacent to $\{0, a_1\}$ and $\{0, a_2\}$, but $\{0, a_1\}$ is not adjacent to $\{0, a_2\}$, a contradiction. So I has only one atom. □

The following example shows that the converse of Lemma 3.7 is not true in general.

Example 3.8. *Suppose that P is a poset in Figure 2. Then it is easy to see that the vertex $I = \{0, a_2\}$, which has only one atom, is not a simplicial vertex.*

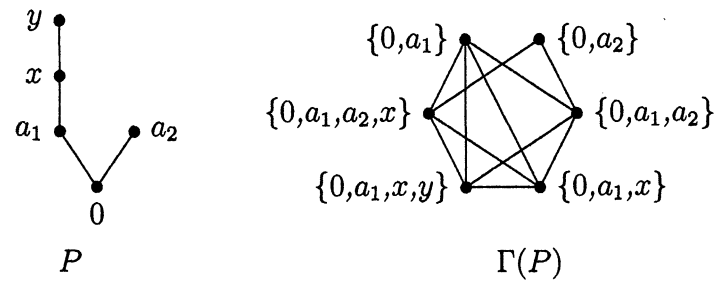


Figure 2

In the next theorem, we provide some conditions under which the converse of Lemma 3.7 holds.

Proposition 3.9. *Let P be a poset and $I \in V(\Gamma(P))$. If I has only one atom and P is one of the following posets in Figure 3, then I is a simplicial vertex.*

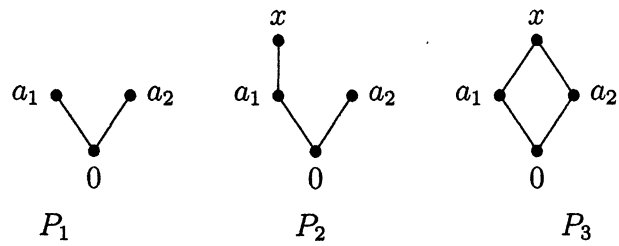


Figure 3

Proof. Assum that P is one of the posets in Figure 3, and $\Gamma(P_1), \Gamma(P_2)$ and $\Gamma(P_3)$ are pictured in Figure 4.

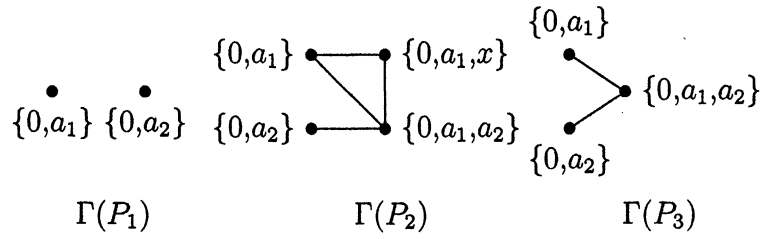


Figure 4

Now, one can easily see that all vertices with one atom in $\Gamma(P_1), \Gamma(P_2)$ and $\Gamma(P_3)$ are simplicial vertices. □

The *diameter* of a connected graph G is the maximum distance between two distinct vertices of G. For any vertex x of a connected graph G, the *eccentricity* of x, denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G. The set of vertices with minimal eccentricity is called the *center* of the graph, denoted by C(G), and this minimum eccentricity value is the *radius* of G, denoted by $r(G)$. (We shall say the diameter and radius of G are zero if G has no edges.)

Notation 3.10. *Let $i_1, i_2, \ldots, i_n$ be integers with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $A(P) = \{a_1, a_2, \ldots, a_n\}$. The notation $U_{i_1 i_2 \ldots i_k}$ stands for the following set:*

$$\{I \trianglelefteq P; \{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\} \subseteq I \text{ and } a_j \notin I, \text{for } j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}\}.$$

Note that all two distinct elements in $U_{i_1 i_2 \ldots i_k}$ are adjacent in $\Gamma(P)$. Also if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $U_{i_1 i_2 \ldots i_k}$ and $U_{j_1 j_2 \ldots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \ldots i_k} \cap U_{j_1 j_2 \ldots j_{k'}} = \emptyset$. Moreover, $V(\Gamma(P)) = \bigcup U_{i_1 i_2 \ldots i_k}$, for all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Also by $1 \cdots \hat{i} \cdots n$ we mean that $1 \cdots i-1\ i+1 \cdots n$.

Lemma 3.11. *Let P be a poset with $|A(P)| = 3$. Then the following statements hold.*

(i) $r(\Gamma(P)) = 3$; and,

(ii) $C(\Gamma(P)) = V(\Gamma(P))$.

Proof. Suppose that I_1 and I_2 are two distinct vertices in $\Gamma(P)$. We have the following situations:

(i) If $I_1, I_2 \in U_i$, for some $1 \leq i \leq 3$, then the vertex I_1 is adjacent to the vertex I_2.

(ii) If $I_1, I_2 \in U_{ij}$, for some $1 \leq i \neq j \leq 3$, then we have the path $I_1 - I_3 - I_2$, for some $I_3 \in I_i$ or $I_3 \in I_j$.

(iii) If $I_1 \in U_{ij}$ and $I_2 \in U_{jk}$, for some $1 \leq i \neq j \neq k \leq 3$, then we have the path $I_1 - I_2 - I_3$, for some $I_3 \in I_j$.

(iv) If $I_1 \in U_i$ and $I_2 \in U_{jk}$, for some $1 \leq i \neq j \neq k \leq 3$, then we have the path $I_1 - I_3 - I_4 - I_2$, for some $I_3 \in I_{ij}$ and $I_4 \in I_j$.

Now, by considering the above situations, we have that, for each $I \in V(\Gamma(P))$, $e(I) = 3$, and so $r(\Gamma(P)) = 3$ and $C(\Gamma(P)) = V(\Gamma(P))$. □

In the sequel of this section, we investigate the eccentricity of the vertices and the center of the graph $\Gamma(P)$, when $|A(P)| \geq 4$. To this end, consider two distinct vertices I_1 and I_2 in $\Gamma(P)$. We have the following situations:

(i) If $I_1, I_2 \in U_i$, for some $1 \leq i \leq n$, then the vertex I_1 is adjacent to the vertex I_2.

(ii) If $I_1, I_2 \in U_{i\ldots j}$, for some $1 \leq i \neq j \leq n$, then we have the path $I_1 - I_3 - I_2$, for some $I_3 \in I_i$ or $I_3 \in I_j$.

(iii) If $I_1 \in U_{i\ldots j}$ and $I_2 \in U_{j\ldots k}$, for some $1 \leq i \neq j \neq k \leq n$, then we have the path $I_1 - I_2 - I_3$, for some $I_3 \in I_j$.

(iv) If $I_1 \in U_i$ and $I_2 \in U_{1\ldots\widehat{i}\ldots j}$, for some $1 \leq i \neq j \leq n$, then we have the path $I_1 - I_3 - I_4 - I_2$, for some $I_3 \in I_{i\ldots j}$ and $I_4 \in I_j$.

(v) If $I_1 \in U_{i\ldots j}$ and $I_2 \in U_{1\ldots\widehat{i}\ldots\widehat{j}\ldots n}$, for some $1 \leq i \neq j \leq n$, then we have the path $I_1 - I_3 - I_4 - I_5 - I_2$, for some $I_3 \in U_i$, $I_4 \in U_{i\ldots k}$ and $I_5 \in U_k$, for some $k \in 1\ldots\widehat{i}\ldots\widehat{j}\ldots n$.

Now, from the above discussion, we have the following proposition.

Proposition 3.12. *Let P be a poset with $|A(P)| \geq 4$ and I be a vertex in $\Gamma(P)$. Then the following statements hold.*

(i) If $I \in U_i$, for some $1 \leq i \leq n$, then $e(I) = 3$. Otherwise, we have $e(I) = 4$.

(ii) $C(\Gamma(P)) = \bigcup_{i=1}^{n} U_i$.

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